Anisotropic turbulence in rotating magnetoconvection

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SUMMARY
Numerical simulations of the 3D MHD-equations that describe rotating magnetoconvection in a Cartesian box have been performed using the code NIRVANA. The characteristics of large-scale quantities like the turbulence intensity and the turbulent heat flux that are caused by the average action of the small-scale fluctuations are computed directly from the fluctuating primitive variables, and the effects of the anisotropy induced by rotation and an external imposed magnetic field are quantified.

The results confirm that the structure of the convection significantly depends on the latitude and the rotation rate as well as on the magnetic field strength and direction. For faster rotation the influence of the Lorentz force on the flow becomes important at lower Els"asser number manifested in the change of the convection pattern and the domination of the azimuthal turbulence intensity. At the pole the turbulence intensity exhibits a slight maximum for an Els"asser number close to unity. No catastrophic quenching occurs as the turbulent quantities are significantly suppressed only for field strengths above the typical values that are assumed to prevail within the fluid outer core of the Earth. In the presence of a horizontal magnetic field the vertical turbulent heat flux increases with increasing field strength so that cooling of the rotating system is facilitated. Horizontal transport of heat is always directed westwards and towards the poles. The latter might be a source of a large-scale meridional flow whereas the first in global simulations would be important in case of non-axisymmetric boundary conditions for the heat flux.

The obtained results could serve as basic clues for the properties of turbulence models that might be applied in future simulations of the geodynamo.

Key words: magnetoconvection – anisotropic turbulence – geodynamo – numerical simulations

1 INTRODUCTION
Anisotropic turbulence is a fundamental feature in many astro- or geophysical systems. Well known realizations are the convective driven motions that occur for example in the solar convection zone (Kitchatinov & R"udiger, 2005) or the fluid flow of liquid iron in the outer core of the Earth – the origin of the Earth’s magnetic field (Braginsky & Roberts, 1995; Hollerbach, 1996; Roberts & Glatzmaier, 2000).

Different from the solar case where either rotation (in the deeper solar convection zone) or the magnetic field (within a sunspot) are dominating, the convection in the Earth’s fluid outer core (FOC) is affected by both, the Lorentz force due to a strong magnetic field and the Coriolis force due to the fast rotation of the Earth. In such rapidly rotating convection driven dynamo the turbulence is subject to three preferred directions defined by magnetic field, rotation axis and gravity/density stratification. The arising anisotropies result in a plate like form of convection cells which are aligned along the rotation axis and elongated in the direction of the dominant magnetic field component (Braginsky & Meytis, 1990; St. Pierre, 1996; Matsushima et al., 1999).

The behavior of the (anisotropic) small-scale flow is of high relevance for numerical examinations of turbulent systems. Global simulations of convection driven dynamos require a very large computational power. Despite of the progresses in computational and numerical techniques that have been made during the recent years, it is still impossible to simulate such systems with a sufficient resolution so that the resolved scale range of the turbulence is rather restricted (Hollerbach, 2003). Furthermore, anisotropy usually is neglected by an implicit assumption of a large eddy simulation (LES) where scalar parameters resemble the turbulent values of the diffusivities. Beside the fact that mostly the diffusivities – for the reason of numerical stability – have to be chosen much larger than even the (often poorly known) “real” turbulent values this oversimplifying assumption describes an isotropic transport of flux and an isotropic dissipation of energy. This is only justified if either the resolution is high enough so that all dynamical important modes are resolved or if no preferred direction originated by external
forces, stratification or boundary conditions exists. Both, in general, is not the case. In order to include anisotropic effects in a more sophisticated way turbulence models have to be considered where e.g. tensorial expressions for the diffusivities take care of the directional dependence of the turbulence and/or the influence of the unresolved scales is parameterized in terms of the resolved large-scale fields.

Tensor coefficients for the viscous and thermal diffusivity of a fast rotating and a strong field model of the Earth’s core have been derived by Phillips & Ivers (2001, 2003). They presented expressions that describe enhanced or suppressed diffusion along the magnetic field lines dominated by the azimuthal component. Although the results were intended for the use in pseudo-spectral geodynamo codes no applications in numerical investigations are available so that the consequences for global simulations remain unknown.

A different approach has been presented by Buffett (2003) who pointed out that a tensorial model might not be sufficient to retrieve the correct behavior of the small-scale turbulence. Additional properties have to be taken into account which was shown by the comparison of direct numerical simulations (DNS) of rotating magnetoconvection with a LES that applies a subgrid-scale (SGS) model based on the self-similarity of the turbulence. As the most important result he demonstrated that the SGS model is able to predict appropriate anisotropic heat- and momentum fluxes. A similar model has been examined by Matsushima (2004, 2005). Although the spatial structures and the temporal behavior could not be reproduced in all details, on average, the results of the LES including the SGS coincided rather well with a fully resolved reference solution. An advanced version of the SGS model that includes the Lorentz- and induction terms was introduced by Matsui & Buffett (2005). They applied a so called non-linear gradient model, an adaption of the similarity model that additionally is based on the local character of the turbulence. A further improvement in relation to the former simulations of magnetoconvection was the validation of this model by performing simulations of dynamo action in a Cartesian box. Again a good agreement was obtained with the reference solution from a higher resolved DNS. However, discrepancies emerged close to the boundaries and in the details of the spatial structure. Further systematic deviations were recognized in average quantities, like kinetic and magnetic energy. The mean values obtained from the LES were situated between the values received from the resolved DNS and the values of a truncated, unresolved DNS.

Beside the mentioned discrepancies, the examined parameter space was rather restricted – especially the behavior of the turbulence at different latitude and the dependence on magnetic field strength as well as direction have only roughly been analyzed. In order to evaluate the functionality of SGS models it might therefore be useful to further investigate the development of a convection driven turbulence under influence of rotation and magnetic field. Subject matter of the present paper are the physical processes which occur on scales that are not resolved in global simulations and the effects that probably are missed by this disregard. A simplified Cartesian system is examined where a convection driven turbulence in a conducting fluid is subject to fast rotation and a (strong) magnetic field. The configuration in principle resembles the most important properties of the Earth’s fluid outer core which are specified by a weak density stratification, fast rotation, a low Mach number flow and a strong magnetic field. Properties like the exact behavior of the temperature gradient, the equation of state or the influence of curvature are assumed to be of secondary importance. Central topic of the investigations is the connection between turbulence characteristics that are caused by the average action of the small-scale fluctuations and the turbulent transport of heat which might essential contributes to the conditions that determine the temperature distribution within the fluid outer core.

2 THE MODEL

2.1 General properties

A detailed description of the applied local model can be found in Giesecke et al. (2005). Fig. 1 shows a sketch of the computational domain, a Cartesian box placed somewhere on a spherical shell at a co-latitude $\theta$. The unit vectors $\hat{x}$, $\hat{y}$, $\hat{z}$ form a right-handed co-rotating Cartesian coordinate system with $\hat{x}$ pointing towards the equator, $\hat{y}$ pointing in the toroidal direction (from west to east) and $\hat{z}$ pointing from the bottom to the top of the box. In global spherical coordinates, $\hat{z}$ represents the radial direction $\hat{r}$ (oriented from inside to outside), $\hat{y}$ the azimuthal direction $\hat{\phi}$ (oriented eastwards) and $\hat{x}$ the meridional direction $\hat{\theta}$ (oriented towards the equator). The angular velocity $\Omega$ in the co-rotating local box coordinate system is given by $\Omega = -\Omega \sin \theta \hat{x} + \Omega \cos \theta \hat{z}$ where $\Omega$ is the angular velocity of the rotating spherical shell. The box with an aspect ratio 8:8:1 is placed at different latitudes on the northern hemisphere of the rotating spherical shell and a standard resolution of 100 $\times$ 100 $\times$ 80 grid points is used in all calculations. The co-latitude angle $\theta$ is varied from $\theta = 0^\circ$ (north pole) to $\theta = 75^\circ$. Unfortunately, it turned out that it is not suggestive to perform simulations close to the equator because for $\theta \gtrsim 75^\circ$ the numerical solutions lead to incon-
2.2 Equations

The MHD-equations for a rotating fluid, including the effects of thermal conduction, compressibility, viscous friction and losses due to magnetic diffusivity, are solved numerically using the code NIRVANA (Ziegler, 1998; Ziegler, 1999). The equations are

\[ \partial_t \rho = -\nabla \cdot (\rho \mathbf{u}), \]

\[ \partial_t (\rho \mathbf{u}) = -\nabla \cdot (\rho \mathbf{u} \mathbf{u}) - \nabla p + \nabla \cdot \sigma + \rho g \]

\[ \quad + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} - 2 \rho \Omega \times \mathbf{u}, \]

\[ \partial_t e = -\nabla \cdot (\rho u \mathbf{u}) - p \nabla \nabla u + \sigma \nabla u + \frac{\partial}{\partial t} \nabla \times \mathbf{B}^2 \]

\[ \quad + \nabla \cdot (\chi \nabla T), \]

\[ \partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B} - \eta \nabla \times \mathbf{B}), \]

with the density \( \rho \), velocity \( \mathbf{u} \), pressure \( p \), magnetic flux density \( \mathbf{B} \), temperature \( T \) and the thermal energy density \( e \). We assume a constant gravitational field \( g = -g \mathbf{\hat{z}} \) within the domain. The viscous stress tensor \( \sigma \) is given by \( \sigma_{ij} = \nu (\partial_i u_j + \partial_j u_i - \frac{2}{3} \nabla \cdot \mathbf{u} \delta_{ij}) \). \( \nu \) denotes the kinematic viscosity and \( \chi \) the thermal conductivity coefficient. The values of \( \chi \), the dynamic viscosity \( \nu_{dyn} = \nu p \) and the magnetic diffusivity \( \eta \) are constant within the box volume. An ideal gas equation of state of assumed is:

\[ p = (\gamma - 1) e = \frac{k}{m \mu_0} \rho T \]

where \( k \) is the Boltzmann constant, \( m \) the atomic mass unit, \( \mu \) the mean molecular weight (\( \mu = 1 \) for all runs) and \( \gamma = c_p/c_v = 5/3 \) is the ratio of the specific heats. The permeability \( \mu_0 \) is given by the vacuum value \( \mu_0 = 4\pi \times 10^{-7} \)VsA\(^{-2}\)m\(^{-1}\).

2.3 Initial state and input parameters

The simulations are started with an initial state that is determined by a hydrostatic equilibrium \( \partial_z p = -\rho g \) and a polytropic temperature distribution \( T = T_0 (\rho/\rho_0)^\Gamma \) in the absence of motions (the subscript 0 refers to values taken at the top boundary of the box). With the equation of state \( \rho(z) = \rho_0 \left( \frac{\frac{d\Gamma}{dT} dT/dz}{T_0} (d - z) \right)^{\frac{1}{\Gamma}} \),

where \( d \) stands for the vertical box extension and the polytropic index \( \Gamma \) is given by \( \Gamma = \ln(1 + \frac{d\Gamma}{dT} dT/dz)/\ln \xi \). The stratification index \( \xi = \rho_{bot}/\rho_{top} \), the temperature \( T_0 \) and the global temperature gradient \( dT/dz \) are prescribed input parameters.

The parameters \( \nu, \chi, \) and \( \eta \) are calculated from the Rayleigh number \( Ra \),

\[ Ra = \frac{\nu g c_p d^4}{\chi \nu} T \left( \frac{dT}{dz} - \frac{g}{c_p} \right) \]

with \( c_p = k (m \nu)^{-1} \gamma (\gamma - 1)^{-1} \) the specific heat at constant pressure, the Prandtl number \( Pr = \nu c_p/\chi \) and the magnetic Prandtl number \( Pm = \nu/\eta \). The basic parameter set used for all simulations that are presented in this paper is given by \( Ra = 10^6, Pr = 0.5 \) and \( Pm = 0.5 \). The diffusivity parameters are scalar quantities assuming that in our local model the resolved scales contain most of the energy, so that the anisotropic transport is described with sufficient accuracy and the remaining non-resolved modes are less important. The rotation rate \( \Omega \) is parameterized by the Taylor number given by

\[ Ta = \frac{4 \Omega^2 d^4}{\nu^2} \]

which is the reciprocal of the Ekman number: \( Ta = E k^{-2} \). Essentially, a rotating system is examined where \( Ta \) is set to a value of \( 10^4 \). A few simulations with a Taylor number \( Ta = 10^4 \) have been performed which demonstrate the significant changes in the behavior of the turbulence as the rotation rate increases.

The magnetic field is expressed by the Elsässer number:

\[ \Lambda = \frac{B^2}{2 \mu_0 \nu p \eta} \]

A represents the relation of the Lorentz force to the Coriolis force which are assumed to be of the same order of magnitude within the FOC. In the linearized equations that describe the onset of convection in this case the Coriolis force can be balanced by the Lorentz force. Therefore there remains no need for balancing these terms with the (extremely small) viscous terms which would constrain the convection to very short length scales. It would be much more difficult to maintain such a flow as it can be seen in the suppression of convective motions in simple rotating convection or non-rotating magnetoconvection. For a detailed description of the underlying mechanism see e.g. Rüdiger & Hollerbach (2004).

2.4 Averaging procedure

Due to the small stratification all considered quantities only weakly depend on the vertical coordinate \( z \) (except close to the upper and the lower boundaries) so that it is justified to characterize the turbulence properties by a volume average defined as

\[ \langle f' \rangle = \frac{1}{N} \sum_{x,y,z} (f_{x,y,z}(t) - \overline{f_z}(t)) \]

where \( N = n_x n_y n_z \) denotes the number of grid cells used for averaging. \( f_{x,y,z}(t) \) represents the value of the quantity \( f \) at a certain grid cell labeled by \( x,y,z \) at a certain time \( t \) and \( \overline{f_z}(t) \) is the horizontal average of the considered quantity. To reduce the undesired influence of the boundaries all volume averages are performed over the inner part of the computational domain (between \( z = 0.25 \) to 0.75). Time averaging is done for periods with no significant changes in the statistical steady convective state. Very long time series have been performed to investigate the essential time span that is necessary for meaningful averages. The convergence
of the time-averaged solutions has been checked by comparing results obtained from averages over different (increasing) periods. To ensure that statistical reasonable results are obtained an averaging period is chosen with a length of at least 20 turnover times \( t_{\text{conv}} = \frac{d}{u_{\text{rms}}} \) with the root mean square velocity \( u_{\text{rms}} \) which is defined as \( u_{\text{rms}} = \sqrt{(u')^2} \).

### 2.5 Boundary Conditions

All quantities are subject to periodic boundary conditions in the horizontal directions. At the top and at the bottom of the computational domain constant values for density and temperature are imposed. The vertical boundary condition for the magnetic field is a perfect conductor condition, and a stress-free boundary condition is adopted for the horizontal components of the velocity \( u_x \) and \( u_y \). Impermeable box walls at the top and the bottom lead to a vanishing \( u_z \) at the vertical boundaries. Table 1 summarizes the boundary conditions and gives the initial values for density and temperature so that the correlations of velocity and temperature with the turbulent transport of heat in the remainder of this paper.

### 3 RESULTS

#### 3.1 Quasi-linear description of the heat transport

The applied parameters result in a pressure dominated (\( \beta = 2 \mu \rho_0 B^2 \gg 1 \)) low Mach number flow (\( \text{Ma} = u_{\text{rms}}/c_s \ll 1 \)) where \( \text{Ma} \) is of the order of \( 10^{-2} \) (depending on rotation rate and field strength: e.g. for \( \Lambda = 1 \) and \( Ta = 10^7 \) the characteristic Mach number is given by \( \text{Ma} \approx 0.03 \)). The corresponding Reynolds numbers \( \text{Re} = u_{\text{rms}} d/\nu \) are of the order of 100. Even in this rather moderate regime the turbulent values of the transport coefficients are larger than the molecular values of the transport coefficients are larger than the molecular transport which describe the overall stratification and the global temperature gradient. For all simulations, temperature and density at the top of the box are scaled to unity, as it is the case for the global temperature gradient \( dT/dz \) and the box height \( d \). A stratification index of \( \xi = \rho_\text{top}/\rho_\text{bot} = 1.1 \) is used.

#### 3.2 Non-rotating magnetoconvection

For a magnetic dominated turbulence without any other preferred direction the tensor \( \chi_{ij} \) is given by

\[
\chi_{ij} = \chi_T \delta_{ij} + \chi_z \frac{\langle B_i B_j \rangle}{\langle B \rangle^2}
\]

(Kitchatinov et al., 1994). For a purely vertical or a purely horizontal field the non-diagonal elements \( \chi_{xz} \) and \( \chi_{yz} \) which are related to the horizontal transport of heat should vanish. However, in principle these components could become non-zero in case of a combined action of a horizontal and a vertical field. Further complexity arises as the coefficients \( \chi_T \) and \( \chi_z \) additionally depend on the magnetic field strength (see e.g. Kitchatinov et al., 1994 and Rüdiger & Kitchatinov, 2000). Eq. (13) does not consider the influence of rotation and it should be only applicable if the Coriolis force is much smaller than the Lorentz force. After the linear theory for the onset of convection in the presence of a vertical magnetic field the critical Rayleigh number scales as \( \text{Ra}_{\text{crit}} \sim H^2 \), where \( H \) represents the Hartmann number

\[
\text{Ha} = \frac{B d}{\sqrt{\mu_0 \rho_0 \eta}}
\]

Therefore the convection is suppressed with increasing field strength which resembles in a reduction of the turbulence intensity \( \langle u_z'^2 \rangle \). Since the preferred wavenumber scales as \( k_{\text{crit}} \sim H^{1/2} \) the typical horizontal size of a convection cell \( \lambda = 1/k_{\text{crit}} \) is reduced with increasing field strength.

### Table 1. Vertical boundary conditions

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>( T )</th>
<th>( u )</th>
<th>( B )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (z = d) )</td>
<td>1</td>
<td>( \partial_x u_z = 0 )</td>
<td>( \partial_x B_z = 0 )</td>
</tr>
<tr>
<td>( \partial_x u_y = 0 )</td>
<td>( \partial_x B_y = 0 )</td>
<td></td>
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<tr>
<td>( u_z = 0 )</td>
<td>( B_z = 0 )</td>
<td></td>
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<tr>
<td>( (z = 0) )</td>
<td>1.1</td>
<td>( \partial_x u_z = 0 )</td>
<td>( \partial_x B_z = 0 )</td>
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<tr>
<td>( \partial_x u_y = 0 )</td>
<td>( \partial_x B_y = 0 )</td>
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<tr>
<td>( u_z = 0 )</td>
<td>( B_z = 0 )</td>
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</table>
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3.3 Convection pattern and anisotropy

The results that are presented in the following are obtained from numerical simulations of rotating magnetoconvection with an imposed horizontal field \((B_y)\) corresponding to a toroidal field in spherical coordinates. The typical structures of the convective motions are visualized in Fig. 2 for \(Ta = 10^6\) (upper row) and for \(Ta = 10^7\) (lower row). The left hand side shows the vertical velocity pattern obtained from a simulation run with \(\Lambda \approx 1\) and the right hand side represents the case \(\Lambda \approx 4\). All plots show a time snapshot of the vertical velocity pattern from a simulation run at \(\theta = 45^\circ\) where the solid iso-surfaces denote \(u_z\) at \(urms\), and the transparent iso-surface denotes the pattern of \(u_z\) at \(0.5urms\). A horizontal cut of the velocity field at a depth \(z = 0.5\) is projected on a plane below the three dimensional box. Blue (dark) tones represent downwards oriented flow and yellow/red (light) colors represent upwards oriented flow. Many features that have been obtained in other simulations of (rotating) convection with and without magnetic field are observable (see e.g. Brummell et al., 1996; Ossendrijver et al., 2001; Ziegler, 2002; Cattaneo et al., 2003; Käpylä et al., 2004). Here we only emphasize the specific characteristics that occur due to the particular conditions given by the weak stratification and the combined action of magnetic field and rotation.

The cell-like objects represented by the volume rendered iso-surfaces of the vertical velocity indicate coherent large-scale structures which span the full vertical extend of the

![Figure 2.](image_url)
domain. These convection cells are continuously formed, rearranged and dissolved leading to a quasi stationary turbulent pattern which is more irregular in the interior of the domain. According to the Taylor-Proudman theorem the convection cells are aligned with the rotation axis. The average horizontal size of the cells strongly depends on the rotation rate as well as the magnetic field strength and direction. The cell size is reduced for an increasing rotation rate and if the horizontal magnetic field exceeds a certain threshold the cells become elongated in direction of the dominant magnetic field component. With respect to Fig. 2 this is only noticeable in the faster rotating case whereas for Ta = 10^6 the convection pattern remains nearly unaffected as the field strength is increased from Λ ≈ 1 to Λ ≈ 4.

Opposite to a configuration with a large density contrast where the structure of the convection pattern consists of isolated, broad warm upflows and narrow network-like, cold and strong downflows, the separation of up- and downflows is less pronounced within in the weakly stratified layer. Instead the upflows form a kind of unconnected single plume-like structures enclosed by a loosely connected broad network of downflows. Between the up- and downflows an extended zone exists where nearly no considerable motions occur. Up- and downflows are of approximately equal amplitude and occupy roughly an equal area in the horizontal plane.

To characterize the anisotropy of the turbulence and to quantify the visual impression obtained from Fig. 2 the average extension of a convection cell is estimated in both horizontal directions. The two-point correlation function Qzz is defined for the direction perpendicular to the imposed magnetic field by

\[ Q_{zz}(\delta x) = \frac{\langle u_z'(x) u_z'(x + \delta x) \rangle}{\langle u_z'(x)^2 \rangle} \tag{15} \]

and for the direction parallel to the field:

\[ Q_{zz}(\delta y) = \frac{\langle u_z'(y) u_z'(y + \delta y) \rangle}{\langle u_z'(y)^2 \rangle}. \tag{16} \]

Fig. 3 shows Qzz(δx) (upper panel) and Qzz(δy) (lower panel) for Λ = 0.1, 1, 10, 100 (solid curves) and corresponding fitting functions (dashed curves) defined by

\[ f(\delta x) = 1 - \frac{\lambda_{corr}^2(\delta x)^2}{\lambda_{corr}^2(\delta y)^2} \tag{17} \]

which are adjusted to be decreasing part of the two-point correlation function. The mean cell size is interpreted as the correlation length \( \lambda_{corr} \), or Taylor microscale which is the characteristic length-scale of the vorticity filaments observed in swirling flows. The shape of the curves of Qzz resemble the decrease of the correlation of the turbulent vertical velocity at two different coordinates with increasing distance between these two points. Qzz can be used as a convenient measure for the average horizontal extension of a convection cell because the motions within a cell are oriented in the same direction so that they are highly correlated whereas no correlation occurs if the distance between the considered two points (given by δx, respectively δy) exceeds the size of the cell.

The behavior of Qzz(δx) which determines the correlation length perpendicular B_y is independent of the imposed field strength (therefore the curves in the upper panel of Fig. 3 are not labeled by Λ). The influence of the magnetic field on the cell structure is only noticeable in the lower panel of Fig. 3 where the correlation function is broadened with increasing field strength.

The resulting correlation lengths \( \lambda_{corr}^x \) – estimated independently for both horizontal directions from Eq. (17) – are shown in Fig. 4. The upper part of the plot denotes the case Ta = 10^6 where the solid (dotted) curve represents \( \lambda_{corr}^x \) (\( \lambda_{corr}^y \)). The lower part shows the same quantities for Ta = 10^7. As already indicated in the upper panel of Fig. 3 \( \lambda_{corr}^x \) is independent from the field strength for both rotation rates so that the extension of the cell in x-direction is only determined by the rotation rate. After the linear theory the preferred length scale for the onset of rotating convection scales as \( Ta^{-1/6} \) (for sufficient fast rotation). Scaling laws for rotating finite amplitude convection have been examined by Stellmach & Hansen (2004). They confirmed the above denoted scaling law for the size of a convection cell by evaluating the preferred wave number \( k = 1/\lambda_{corr}^y \) from the maximum value of the spectral distribution of the kinetic energy. Within the rather restricted range of rotation rates our results reconfirm this scaling for \( \lambda_{corr}^x \):

\[ \frac{\lambda_{corr}^x(Ta = 10^6)}{\lambda_{corr}^x(Ta = 10^7)} \approx 1.5 \approx \left( \frac{10^6}{10^7} \right)^{1/6}. \]
The correlation length parallel to the imposed field, $\lambda_y^{corr}$, resembles the increasing extension of the convection cells in direction of the imposed field. The field strength at which the transition to an anisotropic, elongated cell occurs depends on the rotation rate. In case of $Ta = 10^6$ the transition towards the anisotropic state already becomes noticeable for magnetic fields with $\Lambda \approx 0.1$ whereas for $Ta = 10^6$ we obtain $\lambda_y^{corr} > \lambda_x^{corr}$ only for $\Lambda \gtrsim 1$.

The anisotropic character of the turbulence is also apparent in the behavior of the turbulence intensities $\langle u_i^2 \rangle$. For a quantitative examination the following functions are introduced:

$$A_H = \frac{\langle u_x^2 \rangle - \langle u_y^2 \rangle}{u_{rms}}, \quad A_V = \frac{\langle u_x^2 \rangle + \langle u_y^2 \rangle - 2\langle u_z^2 \rangle}{u_{rms}}$$

(18)

(see also Käpylä et al. (2004) and note the different definition for $A_V$). The anisotropy of the turbulence intensities is described by $A_H$ and $A_V$ in the following way:

- $A_H$: 
  - $< 0$: dominated by turbulence $\perp B_y$
  - $= 0$: horizontal isotropy
  - $> 0$: dominated by turbulence $\parallel B_y$

- $A_V$: 
  - $< 0$: dominated by vertical turbulence
  - $> 0$: dominated by horizontal turbulence

As expected in the presence of a magnetic field the motions $\perp B_y$ are suppressed compared to the motions $\parallel B_y$. The resulting behavior of $A_H$ at $\theta = 0^\circ$ is shown in Fig. 5 which denotes the transition from the horizontal isotropic turbulence to an anisotropic state in dependence of the imposed field. For $Ta = 10^6$ we approximately obtain horizontal isotropy up to field strengths corresponding to $\Lambda \approx 10$ whereas faster rotation results in $A_H > 0$ except for very weak magnetic fields.

The vertical anisotropy, $A_V$, is presented in Fig. 6. Obviously, at the pole the turbulence is dominated by the vertical component independent from the field strength and rotation rate. For $Ta = 10^6$ a slight minimum exists around $\Lambda \approx 10$ which is shifted towards weaker field strength ($\Lambda \approx 1$) in case of faster rotation. For $Ta = 10^7$ the dominance of the vertical component is much better developed around $\Lambda \approx 1$.

For large field strengths the asymptotic behavior of $A_V$ becomes similar for both rotation rates. The dominance of the vertical component is characteristic for the behavior of the turbulence close to the pole and is not maintained for higher co-latitudes (see Sec. 3.5 below).

3.4 Influence of field strength and rotation rate

3.4.1 General behavior of the heat flux

The vertical heat flux is caused by turbulent upflows of warmer fluid and downflows of cooler fluid. Mainly the fastest up- and downflows contribute to the net flux which can be seen in Fig. 7 where time snapshots of the normalized flux $\bar u'_z T'$ evaluated in a horizontal plane at $z = 0.5$ are presented in various scatter plots. Each dot denotes $u'_z T'$ at a certain grid cell in dependence of the local vertical velocity $u'_z$. The sum of all values (divided by the number of data points) represents the horizontal average of $\langle u'_z T' \rangle$ at $z = 0.5$. The four panels show the results from simulations of convection (upper left panel), rotating convection (upper right panel), magnetoconvection (lower left panel) and rotat-
ing magnetoconvection (lower right panel). Qualitatively the behavior is similar for all four rather different states which only distinguish in the amplitude of $u_i^\prime T^\prime$ and of the velocity $u_i^\prime$. The downflows ($u_i^\prime < 0$) and the upflows ($u_i^\prime > 0$) contribute with nearly the same amount to the net flux (although it seems that the downflows are slightly prevailing). This differs from the results for a strongly stratified layer presented by Ziegler (2002) who obtained considerable qualitative (and quantitative) differences between non-rotating and rotating magnetoconvection.

3.4.2 Vertical heat flux and turbulence intensity

From quasi-linear theory a simple relation concerning the thermal conductivity tensor $\chi$ and the one-point correlation tensor $Q_{ij} = \langle u_i^\prime(x, t)u_j^\prime(x, t) \rangle$ exists:

$$\chi_{ij} = \frac{1}{2} \tau_{\text{corr}} Q_{ij} \quad (19)$$

where $\tau_{\text{corr}}$ denotes the correlation time of the turbulence. This is a rather rough estimation which is based on the possibility to express the temperature fluctuations $T^\prime$ through the fluctuating velocity $u^\prime$. Eq. (19) might not be appropriate for a fast rotating system under the simultaneous influence of a magnetic field but it should deliver a general tendency for a relation between the turbulent velocity fluctuations and the (turbulent) thermal conductivity tensor, respectively the normalized heat flux $\hat{F}_{\text{cond}}$.

In Fig. 8 the behavior of the vertical turbulence intensity $\langle u_z^2 \rangle$ (upper panel) is compared with the development of the normalized vertical heat flux, $\langle u_z^\prime T_z^\prime \rangle$ (lower panel). Here, only the differences between the pole (solid line) and a co-latitude at $\theta = 45^\circ$ (dotted line) are pointed out. A more detailed analysis of the angular dependence is given in the next section (3.5). As the most characteristic feature $\langle u_z^2 \rangle$ increases with increasing field strength and exhibits a sharp drop for $\Lambda \gtrsim 2$. The maximum value for $\langle u_z^2 \rangle$ is retrieved around $\Lambda \approx 1$. In this parameter regime (where Lorentz force and Coriolis force are of the same order of magnitude) the turbulence is (slightly) enhanced compared to rotating non-magnetic convection. This feature does not appear at $\theta = 45^\circ$ or for a slower rotation rate. The enhanced turbulence around $\Lambda \approx 1$ has been predicted by Chandrasekhar (1961) who showed that the critical Rayleigh number $R_{\text{crit}}$ is minimal if Lorentz- and Coriolis force are comparable. In case of a fixed (overcritical) Rayleigh number a drop of $R_{\text{crit}}$ leads to a more overcritical convection which results in a stronger driven flow and hence a more vigorous turbulence occurs. Although the linear stability analysis of Chandrasekhar (1961) has been performed for a vertical oriented field ($B_z$), however, the general trend might be also true for a horizontal field. The reduction of $R_{\text{crit}}$ increases with increasing rotation rate so that the effect should become more obvious for faster rotating systems. One could therefore expect that the maximum structure of $\langle u_z^2 \rangle$ will be more dominant for even faster rotation, a result that e.g. has been obtained by Stellmach & Hansen (2004). However, they also obtained hints, that this effect – predicted from a linear stability analysis – vanishes for stronger driven flows where the non-linearities dominate the final state.
In accordance with the behavior of \( \langle u_x'^2 \rangle \) the normalized vertical heat flux increases with the field strength below \( \Lambda \approx 1 \) but \( \langle u_y'^2 \rangle \) remains at a constant high level even for strong fields (opposite to \( \langle u_x'^2 \rangle \)). Although the fluctuating fluid motions are inhibited in the presence of a strong magnetic field, this is obviously not the case for the temperature fluctuations, so that the correlation of these two quantities is not suppressed with increasing field strength. This feature is not observable at a higher co-latitude where the \( \langle u_x'^2 \rangle \) exhibits a maximum around \( \Lambda \approx 0.5 \) and is suppressed for stronger field.

A second distinctive feature is the crossing of the curves for \( \theta = 0^\circ \) and \( \theta = 45^\circ \). For weak fields the turbulence intensity and the vertical heat flux at \( \theta = 45^\circ \) are larger than at the pole whereas both quantities behave the other way round for a field strength above \( \Lambda \approx 0.1...0.5 \). Therefore, the introduction of a horizontal magnetic field in a rotating and convecting system favors the turbulence and heat transport at the pole.

In order to investigate the influence of the orientation of the imposed magnetic field some simulation runs have been performed involving a vertical magnetic field \( B_z \) (note, that in this case the magnetic boundary conditions have to be changed to \( B_z = B_y = 0 \) and \( \partial_z B_z = 0 \)). The behavior of the vertical turbulence intensity and the heat flux at the pole and at \( \theta = 45^\circ \) are shown in Fig. 9. Turbulence intensity and \( \langle u_y'^2 \rangle \) are slightly larger at \( 45^\circ \) and both quantities exhibit nearly the same dependence on the field strength indicating that the correlation time defined in Eq. (19) is nearly independent of the field. A very effective suppression occurs for \( \Lambda \gtrsim 2 \), and for \( \Lambda \gtrsim 20 \) no vertical transport of heat through turbulent motions is observable. Such high magnetic field strength completely inhibits convection because in the present configuration the critical Rayleigh number probably is larger than the applied Rayleigh number.

### 3.4.3 Horizontal heat fluxes and Reynolds stresses

The horizontal components of the normalized heat flux \( \langle u_x'^2 \rangle \) and \( \langle u_y'^2 \rangle \) and the corresponding Reynolds stresses \( Q_{xx} \) and \( Q_{xy} \) vanish at the poles. This expected behavior is nearly perfectly reflected in the numerical results and therefore the dependence on the field strength is only considered at \( \theta = 45^\circ \). Fig. 10 shows \( Q_{xx} \) and \( Q_{xy} \) (upper panel) in comparison with \( \langle u_x'^2 \rangle \) and \( \langle u_y'^2 \rangle \) (lower panel). Note, that now again a horizontal magnetic field \( B_z \) is imposed. In all cases the considered quantities are negative. Therefore, the latitudinal (or meridional) component, \( \langle u_y'^2 \rangle \), describes a transport of heat towards the (north-)pole. The absolute value increases with increasing field strength so that the relation between \( \langle u_y'^2 \rangle \) and \( Q_{xx} \) cannot be described by Eq. (19) since the absolute value of \( Q_{xx} \) decreases with increasing field strength.

The \( y \)-component of the heat flux corresponds to an azimuthal transport of heat in a spherical geometry, \( \langle u_y'^2 \rangle \), as it is also always negative, describes a westward directed transport of heat. Similar to the meridional component, the absolute value increases with increasing field strength, however, a distinct minimum is observable around \( \Lambda \approx 10 \). For very strong fields \( \langle u_y'^2 \rangle \) might vanish as it is expected from Eq. (13) and the behavior of the azimuthal heat flux roughly resembles the development of \( \langle u_y'u_z' \rangle \), although there exists a small offset of the location of the minimum of \( Q_{xy} \) towards weaker magnetic fields.

A negative azimuthal heat flux has also been obtained by Rüdiger et al. (2005) for solar like rotating convection, but as they only investigated an axisymmetric configuration this result remains without any consequences for their considerations.

### 3.5 Angular dependence

#### 3.5.1 Transition from vertical to horizontal dominated turbulence

As expected and indicated in the plot of the horizontal anisotropy shown in the upper panel of Fig. 5, in general the suppression of the motions perpendicular to the imposed magnetic field is stronger than the suppression of the motions parallel to the field (if the field strength is sufficient large). For a strong field we also retrieve \( \langle u_x'^2 \rangle > \langle u_y'^2 \rangle \) independent from the latitude angle and in all cases \( \langle u_x'^2 \rangle \) towards the equator exceeds the intensities at the pole. The increase of the horizontal intensities towards the equator leads to a significant change in the characteristics of the turbulence. Except for \( \Lambda \approx 1 \) the vertical component is dominating at the pole, whereas towards the equator the turbulence is dominated by the horizontal components (see

![Figure 9. Vertical heat flux for an vertical oriented magnetic field (Bz).](image-url)
Note, that it might be possible that around $\Lambda \approx 1$ and normalized heat flux the turbulent heat conductivity tensor $\chi_{ij}$ can be approximated by an expression of the form given by equation (22). However, this is only possible if $\Lambda \approx 0.1$ and $\Lambda \approx 1$ the absolute value exhibits a maximum around $\theta = 45^\circ$ and in both cases $\langle |u'_y T'\rangle \rangle$ can be described by a function $\propto \sin \theta \cos \theta$ as indicated by equation (21). This approximation fails for strong fields ($\Lambda \approx 10$) where $|\langle u'_y T'\rangle|$ increases towards the equator.

The angular dependence of the Reynolds stresses and the normalized turbulent heat flux are shown in Fig. 12 (left hand side: $Q_{xz}$; right hand side $\langle u'_y T'\rangle$). The solid (dotted, dashed) lines denote the results for $\Lambda = 0.1(1,10)$. In all cases the meridional component $\langle u'_y T'\rangle$ is negative. For $\Lambda \approx 0.1$ and $\Lambda \approx 1$ the absolute value increases with increasing $\theta$ (and $\Lambda$) whereas for strong fields ($\Lambda \approx 10$) $\langle u'_y T'\rangle$ decreases towards the equator.

The normalized vertical heat flux is always positive thus heat is transported outwards. For $\Lambda \approx 0.1$ and $\Lambda \approx 1$ $\langle u'_z T'\rangle$ increases towards the equator and in principle could be approximated by an expression of the form given by equation (23). However, this is only possible if $\chi_{||}$ is negative which would be in contradiction with the results for the meridional heat flux which require $\chi_{||} > 0$. For $\Lambda \lesssim 1$ the vertical heat flux exhibits a minimum at about $\theta \approx 30^\circ$ whereas for stronger fields ($\Lambda \gtrsim 10$), $\langle u'_z T'\rangle$ decreases towards the equator.

For $\Lambda \approx 0.1$ and $\Lambda \approx 1$ the components of the heat flux and $Q_{xz}$ are roughly in agreement with Eq. (19) (apart from the behavior of $Q_{xz}$ towards the equator). For stronger fields the increase of the absolute values of the horizontal components of the heat flux cannot be explained in such a simple form.

### 3.5.2 Heat flux

In case of a rotating turbulence without any other preferred direction the turbulent heat conductivity tensor $\chi_{ij}$ is given by

$$\chi_{ij} = \chi' \delta_{ij} + \chi_{||} \frac{\Omega_i \Omega_j}{2\Omega} + \chi_{\epsilon ij} \Omega_p$$

(see e.g. Kitchatinov et al., 1994). In the applied Cartesian system where we are restricted to a vertical large scale temperature gradient the following three elements of $\chi_{ij}$ can be computed:

$$\chi_{xz} = -\chi_{||} \sin \theta \cos \theta$$

(21)

$$\chi_{yz} = -\chi_{||} \Omega_2 \sin \theta$$

(22)

$$\chi_{zx} = \chi_T + \chi_{||} \cos^2 \theta$$

(23)

These expressions neglect the influence of the magnetic field and might therefore be valid only for small values of $\Lambda$.

### Figure 10

Horizontal Reynolds stresses $\langle u'_x u'_y \rangle$ (upper panel) and normalized heat flux $\langle u'_y T' \rangle$ (lower panel) in dependence of the imposed magnetic field strength. $\theta = 45^\circ$ and $T_0 = 10^7$.

### Figure 11

Transition from vertical dominated turbulence to horizontal dominated turbulence.
Anisotropic turbulence in rotating magnetoconvection

Figure 12. Left side: Reynolds stresses $Q_{xz}, Q_{yz}$ and $Q_{zz} = u'^2_z$. Right side: Heat fluxes $\langle u'_x T' \rangle$, $\langle u'_y T' \rangle$ and $\langle u'_z T' \rangle$. Both quantities are shown in dependence of the co-latitude $\theta$ for $Ta = 10^7$. The solid (dashed, dotted) curve denotes $\Lambda = 0(1, 10)$.

$30^\circ$ and the intensity towards the equator is larger than at the pole. The difference between the turbulence intensities at the equator and at the pole is reduced with increasing magnetic field and for strong magnetic fields ($\Lambda \approx 10$) $\langle u'^2_z \rangle$ is even slightly larger at the pole. Note that for $\Lambda \approx 1$ the enhanced intensity shown in Fig. 8 and corresponding to a more vigorous convection due to the drop in the critical Rayleigh number, only exists close the pole. At higher latitudes, $\langle u'^2_z \rangle$ is simply reduced with increasing field strength as it has already been shown on the lower left panel in in Fig. 8.

Käpylä et al. (2004) calculated the vertical and the meridional heat flux for rotating convection in a strong stratified layer for a various number of rotation rates. In all cases they report the maximum of the vertical heat flux at the equator and “a tendency” for the minimum at a co-latitude $\theta \approx 30^\circ...45^\circ$. These results are confirmed by our calculations up to $\Lambda \approx 1$. The results of Käpylä et al. (2004) for the meridional heat flux are incoherent, showing different
parameter ranges with positive and negative values. Within our examination we only obtain positive values for \((u'_xT')\) for rather slow rotation rates and close to the equator. In all other cases we obtained a negative meridional heat flux.

4 DISCUSSION AND CONCLUSIONS

Rotating magnetoconvection has been examined for a wide range of parameters and due to the varieties of the results it is nearly impossible to describe the outcome uniformly. The structure of the convective flow change its characteristic properties considerably in dependence of the applied parameters. For faster rotation the effects of the magnetic field on the fluid flow via the Lorentz force becomes significant at a much lower Elsässer number \(\Lambda\) which supports the necessity of high rotation rates to resemble the dynamics of the fluid flow in a realistic manner. As the most important consequence, convection in a fast rotating system should occur in extremely thin sheet like structures. In the presented simulations the vertical extension of the cells is fixed by the vertical boundary conditions so that the average size of the convection cells in \(z\)-direction is extended over the whole box domain. It is hardly believable that such structures are stable over the largest possible scales given by the size of the fluid outer core so that the “real” vertical structure of the convection remains dubious and might only be obtained from unimaginable high resolved global simulations.

As an example of a (turbulent) diffusivity quantity the turbulent heat flux has been examined. It could be shown that all components strongly depend on the location, respectively the co-latitude and the existing magnetic field. Interestingly, all components of the heat flux exhibit a distinct enhancement at field strengths that are assumed to be characteristic in the Earth’s core. As the most remarkable characteristic the vertical heat flux increases with increasing magnetic field strength (if the rotation is fast enough) and remains at a high level even for very strong magnetic fields. Thus cooling of the Earth’s interior is enhanced by the presence of a dominant azimuthal field component. Comparisons of the results from non-rotating magnetoconvection (see Sec. 3.2) with simulations that involve rotation and a horizontal (Fig. 8) respectively a vertical oriented magnetic field (Fig. 9) show that an enhancement of the vertical heat flux at high field strength only occurs in combination of a horizontal oriented magnetic field, a sufficient fast rotation rate and close to the north-pole.

Likewise of interest is the behavior of the meridional heat flux which is always negative, indicating that heat is transported towards the poles. From comparison with slower rotating magnetoconvection we know that the rotational quenching of \((u'_xT')\) is weaker than for \((u'_yT')\) and \((u'_zT')\), so that it might be possible that in a fast rotating system a significant amount of heat is transported towards the poles. As a consequence the poles should be warmer than the equator resulting in a non-radial oriented large-scale temperature gradient which acts as a source for a meridional large-scale flow. Such a meridional flow – if strong enough – affects the column like large-scale structure of the convective motions and might be very important for the understanding of the dynamo mechanism in the Earth’s core (See e.g. Sarson & Jones 1999, who identified a fluctuating meridional flow as an explanation for the occurrence of dipole reversals in a simplified 2D geodynamo-model). However, the circumstances within a rotating spherical shell are more complicated as they appear in the presented Cartesian model where relevant characteristics have not been concerned for simplicity. Beside the effects of curvature or compositional convection which have completely been ignored within the present study e.g. it is unknown if the horizontal field component is dominating at the pole (if it is dominating at all). Furthermore, the distribution of the temperature at the outer boundary depends on the heat flow that is usually prescribed as a boundary condition at the core mantle boundary (CMB). Seismological measurements indicate a non-uniform heat flow at the CMB, whose influence has been examined in numerical simulations of e.g. Olson & Christensen (2002). The lateral and longitudinal variations of the boundary heat-flux affect the local behavior of the fluid flow: Convection is enhanced (inhibited) where the boundary heat-flow is high (low). In such case where the boundary conditions at the top of the fluid core contain non-axisymmetric contributions the azimuthal heat flux becomes important. The azimuthal heat flux is always negative and increases significantly towards the equator. This leads to a latitude dependent westward transport of heat which should interact with the constraints given by the inhomogeneous thermal conditions at the CMB. Again the consequences of such behavior could only be examined by a global simulation.

The obtained results yield a rather complicated picture of the interactions of the diverse physical processes whose consequences are not immediately (if at all) obvious. It might therefore be difficult to adopt the outcome in any global geodynamo model in particular because a “simple” tensor model – as already mentioned – probably is not sufficient to parameterize the correct behavior of the small-scale turbulence. Furthermore, the deduced properties for example for the turbulent heat conductivity tensor appear too manifold and complex which prevents from a simple parameterization. However, it has been shown, that small-scale processes contribute to processes that determine the large-scale fields and it is therefore recommendable to consider their effects as accurate as possible. The present examination might serve as constraints for future SGS-models which should resemble the illustrated properties of anisotropic rotating magnetoconvection, but they will not replace the necessity to perform very high resolved global simulations.

REFERENCES

Anisotropic turbulence in rotating magnetoconvection


