

Mean-field electrodynamics: critical analysis of various analytical approaches to the mean electromotive force

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There are various analytical approaches to the mean electromotive force $\mathcal{E} = \langle \mathbf{u} \times \mathbf{b} \rangle$ crucial in mean-field electrodynamics, with \mathbf{u} and \mathbf{b} being velocity and magnetic field fluctuations. In most cases the traditional approach, restricted to the second-order correlation approximation, has been used. Its validity is only guaranteed for a range of conditions, which is narrow in view of many applications, e.g., in astrophysics. With the intention to have a wider range of applicability, other approaches have been proposed which make use of the so-called τ -approximation, reducing correlations of third order in \mathbf{u} and \mathbf{b} to such of second order. After explaining some basic features of the traditional approach a critical analysis of the approaches of that kind is given. It is shown that they lead in some cases to results which are in clear conflict with those of the traditional approach. It is argued that this indicates shortcomings of the τ -approaches and poses serious restrictions to their applicability. These shortcomings do not result from the basic assumption of the τ -approximation. Instead, they seem to originate in some simplifications made in order to derive \mathcal{E} without really solving the equations governing \mathbf{u} and \mathbf{b} . A starting point for a new approach is described which avoids the conflict.

Keywords: Mean-field magnetohydrodynamics; Mean electromotive force; Second-order correlation approximation; τ -approaches

1. Introduction

In mean-field electrodynamics, the mean electromotive force $\mathcal{E} = \langle \mathbf{u} \times \mathbf{b} \rangle$ due to the fluctuations \mathbf{u} and \mathbf{b} of the fluid velocity and the magnetic field plays a crucial role (Krause and Rädler 1971, 1980, Moffatt 1978, Rädler 2000). A central problem is the determination of \mathcal{E} for a given motion as a functional of the mean magnetic field. Various methods have been used for that.

One approach, which we call “traditional approach” or “approach (i)” in the following, was established together with mean-field electrodynamics at all (Krause and Rädler 1971, 1980). Most of the calculations of \mathcal{E} have been done on

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this basis using the so-called second-order correlation approximation (SOCA) or, what means the same, first-order smoothing approximation (FOSA). This approximation in its original form, that is, applied in the case of a purely hydrodynamic background turbulence, ignores all higher than second-order correlations in the fluctuations \mathbf{u} of the velocity field (see section 3.1). It can be justified only in cases in which these fluctuations are not too large. The simple sufficient conditions usually given for its validity are rather narrow in view of astrophysical applications. Basically it is possible to proceed to higher-order approximations but this requires tremendous efforts and has been done so far only in a few simple cases (Nicklaus and Stix 1988, Carvalho 1992, 1994, Rädler *et al.* 1997, see also section 5.1). A slight modification of the second-order correlation approximation applies also to the case of a magnetohydrodynamic background turbulence (see section 3.2).

In some more recent investigations (which are cited below) other approaches are used, which rely in a sense on the τ -approximation of turbulence theory (Orszag 1970) and are called “ τ -approaches” or “approaches (ii)” in the following. They go in so far beyond the second-order correlation approximation as they also consider higher-order correlations, which are then in the sense of a closure expressed by second-order ones. In approach (i) the relevant equations are simplified by a well-defined approximation and then really solved, and \mathcal{E} is calculated with these solutions. In the approaches (ii), a relation for \mathcal{E} is deduced from the original equations, but without really solving them. Instead, in order to get manageable results, assumptions on the connection of some of the occurring quantities with \mathcal{E} are introduced. The final result for \mathcal{E} is to a large extent determined by these assumptions. The approaches (ii) cover from the very beginning also the case of a magnetohydrodynamic background turbulence.

The main purpose of this article is a critical analysis of the approaches (ii). Each step of approach (i) can be justified by the underlying induction equation or, in the case of a magnetohydrodynamic background turbulence, induction equation and momentum balance. Clear, at least sufficient, conditions for the applicability of the second-order approximation can be given. There is no doubt in the correctness of its results in the so defined range of applicability. It seems reasonable to assume, and we do so in this article, that there is at least some overlap of the ranges of applicability of the approaches (i) and (ii). We have then to require that in these overlapping ranges the results of both approaches coincide. Simple versions of approaches (ii) as used by Vainshtein and Kichatinov (1983), Blackman and Field (2002) and Brandenburg and Subramanian (2005b), called “simple τ -approach” or “approach (iia)” in the following, deliver results which do not in all cases satisfy this requirement. As we will show below, the more sophisticated version used in the articles by Rädler *et al.* (2003) and by Rogachevskii and Kleeorin (2003, 2004), called “spectral τ -approach” or “approach (iib)”, does not satisfy this requirement, too. We have to conclude that the approaches (ii) are not in full accordance with the basic equations mentioned. Therefore the results cannot be taken for granted. We will propose a starting point for an alternative approach, which avoids conflicts with approach (i).

In section 2, we define the frame of our considerations and deliver the basic equations. In section 3, we recall the fundamentals of approach (i) and review some of its basic results. In section 4, we explain the approaches (iia) and (iib), derive a few results, restricting attention to the simple case of a non-rotating fluid, and pinpoint shortcomings of these approaches and deviations of the results from those

of approach (i). In section 5, we explain a proposal for the alternative approach mentioned. Finally in section 6, we summarize our findings.

2. The mean-field concept in magnetofluidynamics

Let us first define the frame of our considerations. We want to explain essential features of the various approaches to the mean electromotive force \mathcal{E} but do not strive to a high level of generality.

2.1. Mean-field electrodynamics

Let us consider the magnetic field \mathbf{B} in an electrically conducting moving fluid. We assume that it is governed by the induction equation

$$\partial_t \mathbf{B} = \eta \nabla^2 \mathbf{B} + \nabla \times (\mathbf{U} \times \mathbf{B}), \quad \nabla \cdot \mathbf{B} = 0. \quad (1)$$

Here \mathbf{U} means the velocity of the fluid and η its magnetic diffusivity, assumed to be constant. Until further notice we consider the fluid motion, that is \mathbf{U} , as given.

We assume that the motion and therefore also the magnetic field shows irregular, e.g. turbulent, features. Any field F of this type is split into a mean field \overline{F} and a “fluctuating” field f , that is, $F = \overline{F} + f$. The mean field \overline{F} is defined as an average of F . It is assumed that the averaging procedure satisfies the Reynolds averaging rules. Alternatively to the notation \overline{F} , we use also $\langle F \rangle$ in the following.

Averaging the induction equation (1), we obtain the mean-field induction equation

$$\partial_t \overline{\mathbf{B}} = \eta \nabla^2 \overline{\mathbf{B}} + \nabla \times (\overline{\mathbf{U}} \times \overline{\mathbf{B}} + \mathcal{E}), \quad \nabla \cdot \overline{\mathbf{B}} = 0, \quad (2)$$

with the mean electromotive force \mathcal{E} given by

$$\mathcal{E} = \overline{\mathbf{u} \times \mathbf{b}}. \quad (3)$$

Elaboration of mean-field electrodynamics means studying the properties of this quantity. For this purpose we rely on the equation for the fluctuations \mathbf{b} which follows from (1) and (2),

$$\partial_t \mathbf{b} = \eta \nabla^2 \mathbf{b} + \nabla \times (\overline{\mathbf{U}} \times \mathbf{b} + \mathbf{u} \times \overline{\mathbf{B}}) + \mathbf{G}, \quad \nabla \cdot \mathbf{b} = 0. \quad (4)$$

Here \mathbf{G} stands for a term of second order in \mathbf{u} and \mathbf{b} ,

$$\mathbf{G} = \nabla \times (\mathbf{u} \times \mathbf{b})', \quad (5)$$

with $(\mathbf{u} \times \mathbf{b})' = \mathbf{u} \times \mathbf{b} - \overline{\mathbf{u} \times \mathbf{b}}$.

As can be concluded from (3) and (4) the mean electromotive force \mathcal{E} depends, apart from initial and boundary conditions for \mathbf{b} , on $\overline{\mathbf{U}}$, \mathbf{u} and $\overline{\mathbf{B}}$. More precisely, \mathcal{E} is a functional of these quantities in the sense that \mathcal{E} in a given point in space and time depends on $\overline{\mathbf{U}}$, \mathbf{u} and $\overline{\mathbf{B}}$ also in other space points and at past times. This functional is linear in $\overline{\mathbf{B}}$. For most of the applications it is reasonable to assume that it is in

addition local in the sense that \mathcal{E} , in a point in space and time, depends only on $\bar{\mathbf{U}}$, \mathbf{u} and $\bar{\mathbf{B}}$ in certain surroundings of this point. Likewise in many applications the variations of $\bar{\mathbf{B}}$ in space and time are sufficiently weak so that the behavior of $\bar{\mathbf{B}}$ inside the relevant surroundings of a given point can be represented by $\bar{\mathbf{B}}$ and its first spatial derivatives in this point. Then we have

$$\mathcal{E}_i = \mathcal{E}_i^{(0)} + a_{ij}\bar{B}_j + b_{ijk}\frac{\partial\bar{B}_j}{\partial x_k}, \quad (6)$$

where the quantities $\mathcal{E}_i^{(0)}$, a_{ij} and b_{ijk} are functionals of $\bar{\mathbf{U}}$ and \mathbf{u} in the above sense but do not depend on $\bar{\mathbf{B}}$.

Assume for a simple example that there is no mean motion, $\bar{\mathbf{U}} = \mathbf{0}$, and that \mathbf{u} corresponds to a homogeneous isotropic, not necessarily mirror-symmetric turbulence. Since no isotropic vector exists, $\mathcal{E}_i^{(0)}$ must be equal to zero. Further a_{ij} and b_{ijk} have to be isotropic tensors, $a_{ij} = \alpha\delta_{ij}$ and $b_{ijk} = \beta\epsilon_{ijk}$, with factors α and β being averaged quantities depending on \mathbf{u} . Consequently we have

$$\mathcal{E} = \alpha\bar{\mathbf{B}} - \beta\nabla \times \bar{\mathbf{B}}. \quad (7)$$

For another simple example we retain $\bar{\mathbf{U}} = \mathbf{0}$ and assume that \mathbf{u} corresponds to an inhomogeneous turbulence in a rotating system, that is, under the influence of a Coriolis force. More precisely, \mathbf{u} deviates from a homogeneous isotropic mirror-symmetric turbulence only by an inhomogeneity and therefore an anisotropy described by a vector \mathbf{g} parallel to the intensity gradient, and by the anisotropy and the deviation from mirror-symmetry described by the angular velocity $\boldsymbol{\Omega}$ responsible for the Coriolis force. Then $\mathcal{E}_i^{(0)}$ needs no longer to be zero, and a_{ij} and b_{ijk} are no longer purely isotropic tensors. For the sake of simplicity, we nevertheless ignore here $\mathcal{E}_i^{(0)}$, referring to the comments in sections 3, 3.1.1, 3.1.5, 3.2.1 and 5.1. We further assume that the influences of \mathbf{g} and $\boldsymbol{\Omega}$ on the turbulence are weak enough so that \mathcal{E} is linear in these quantities. Considering then the symmetry properties of the basic equations (see, e.g., Krause and Rädler 1971, 1980) we have

$$\begin{aligned} \mathcal{E} = & -\gamma\mathbf{g} \times \bar{\mathbf{B}} - \alpha_1(\mathbf{g} \cdot \boldsymbol{\Omega})\bar{\mathbf{B}} - \alpha_2(\mathbf{g} \cdot \bar{\mathbf{B}})\boldsymbol{\Omega} - \alpha_3(\boldsymbol{\Omega} \cdot \bar{\mathbf{B}})\mathbf{g} \\ & - \beta\nabla \times \bar{\mathbf{B}} - \delta_1(\boldsymbol{\Omega} \cdot \nabla)\bar{\mathbf{B}} - \delta_2\nabla(\boldsymbol{\Omega} \cdot \bar{\mathbf{B}}), \end{aligned} \quad (8)$$

where the coefficients $\gamma, \alpha_1, \dots, \delta_2$ are determined by \mathbf{u} only. (The choice of the signs in (8), which follows some other articles, is however unimportant at this place.)

Given the structures of \mathcal{E} as in these examples, the only remaining task is the calculation of coefficients like α, β , etc. The approximations we want to discuss in this article concern only these calculations.

2.2. Dynamical aspects

Later we will relax the assumption that the fluid motion, that is \mathbf{U} , is given. In preparation to this, we provide here the relevant relations. Restricting our attention to an incompressible fluid we assume that \mathbf{U} is governed by the momentum balance,

$$\begin{aligned} \rho(\partial_t\mathbf{U} + (\mathbf{U} \cdot \nabla)\mathbf{U}) = & -\nabla P + \rho\nu\nabla^2\mathbf{U} - 2\rho\boldsymbol{\Omega} \times \mathbf{U} \\ & + \frac{1}{\mu}(\nabla \times \mathbf{B}) \times \mathbf{B} + \rho\mathbf{F}, \quad \nabla \cdot \mathbf{U} = 0. \end{aligned} \quad (9)$$

Here ρ is the mass density, considered to be constant, P the hydrodynamic (including centrifugal) pressure, ν the kinematic viscosity, taken as constant too, and \mathbf{F} means an external force. A rotating frame of reference is assumed, with $\boldsymbol{\Omega}$ being the angular velocity that defines the Coriolis force.

Taking the average of equation (9) we obtain

$$\begin{aligned} \rho(\partial_t \bar{\mathbf{U}} + (\bar{\mathbf{U}} \cdot \nabla) \bar{\mathbf{U}}) &= -\nabla \bar{P} + \rho \nu \nabla^2 \bar{\mathbf{U}} - 2\rho \boldsymbol{\Omega} \times \bar{\mathbf{U}} \\ &+ \frac{1}{\mu} (\nabla \times \bar{\mathbf{B}}) \times \bar{\mathbf{B}} + \rho (\bar{\mathbf{F}} + \mathcal{F}), \quad \nabla \cdot \bar{\mathbf{U}} = 0, \end{aligned} \quad (10)$$

with a mean ponderomotive force \mathcal{F} given by

$$\mathcal{F} = -\overline{(\mathbf{u} \cdot \nabla) \mathbf{u}} + \frac{1}{\mu \rho} \overline{(\nabla \times \mathbf{b}) \times \mathbf{b}}. \quad (11)$$

In view of both \mathcal{E} and \mathcal{F} , the equations for the fluctuations \mathbf{u} are of interest. They can be concluded from (9) and (10),

$$\begin{aligned} \partial_t \mathbf{u} + (\bar{\mathbf{U}} \cdot \nabla) \mathbf{u} + (\mathbf{u} \cdot \nabla) \bar{\mathbf{U}} \\ = -\frac{1}{\rho} \nabla(p + \frac{1}{\mu} \bar{\mathbf{B}} \cdot \mathbf{b}) + \nu \nabla^2 \mathbf{u} - 2 \boldsymbol{\Omega} \times \mathbf{u} \\ + \frac{1}{\mu \rho} ((\bar{\mathbf{B}} \cdot \nabla) \mathbf{b} + (\mathbf{b} \cdot \nabla) \bar{\mathbf{B}}) + \mathbf{f} + \mathbf{T}, \quad \nabla \cdot \mathbf{u} = 0. \end{aligned} \quad (12)$$

\mathbf{T} stands for the terms of second order in \mathbf{u} and \mathbf{b} ,

$$\mathbf{T} = -((\mathbf{u} \cdot \nabla) \mathbf{u})' - \frac{1}{2\mu\rho} \nabla(\mathbf{b}^2)' + \frac{1}{\mu\rho} ((\mathbf{b} \cdot \nabla) \mathbf{b})', \quad (13)$$

where $((\mathbf{u} \cdot \nabla) \mathbf{u})' = (\mathbf{u} \cdot \nabla) \mathbf{u} - \overline{(\mathbf{u} \cdot \nabla) \mathbf{u}}$ etc.

For multiple purposes it is necessary to eliminate the pressure term $(1/\rho) \nabla(p + (1/\mu) \bar{\mathbf{B}} \cdot \mathbf{b})$ in (12). Assuming an infinitely extended fluid, we find

$$\partial_t \mathbf{u} = \nu \nabla^2 \mathbf{u} + \tilde{\mathbf{R}} + \tilde{\mathbf{L}} + \tilde{\mathbf{U}} + \tilde{\mathbf{f}} + \tilde{\mathbf{T}}, \quad \nabla \cdot \mathbf{u} = 0, \quad (14)$$

with

$$\begin{aligned} \tilde{\mathbf{R}} &= 2(\boldsymbol{\Omega} \cdot \nabla) \psi, \quad \psi(\mathbf{x}) = \frac{1}{4\pi} \int_{\infty} \frac{\nabla' \times \mathbf{u}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3 x' \\ \tilde{\mathbf{L}} &= \frac{1}{\mu\rho} ((\bar{\mathbf{B}} \cdot \nabla) \mathbf{b} + (\mathbf{b} \cdot \nabla) \bar{\mathbf{B}} + \nabla M), \quad M(\mathbf{x}) = \frac{1}{2\pi} \int_{\infty} \frac{\partial \bar{B}_i(\mathbf{x}') / \partial x'_j \cdot \partial b_j(\mathbf{x}') / \partial x'_i}{|\mathbf{x} - \mathbf{x}'|} d^3 x' \\ \tilde{\mathbf{U}} &= -(\bar{\mathbf{U}} \cdot \nabla) \mathbf{u} - (\mathbf{u} \cdot \nabla) \bar{\mathbf{U}} - \nabla V, \quad V(\mathbf{x}) = \frac{1}{2\pi} \int_{\infty} \frac{\partial \bar{U}_i(\mathbf{x}') / \partial x'_j \cdot \partial u_j(\mathbf{x}') / \partial x'_i}{|\mathbf{x} - \mathbf{x}'|} d^3 x' \\ \tilde{\mathbf{f}} &= \mathbf{f} + \nabla g, \quad g(\mathbf{x}) = \frac{1}{4\pi} \int_{\infty} \frac{\nabla' \cdot \mathbf{f}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3 x' \\ \tilde{\mathbf{T}} &= \mathbf{T} + \nabla W, \quad W(\mathbf{x}) = \frac{1}{4\pi} \int_{\infty} \frac{\nabla' \cdot \mathbf{T}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3 x'. \end{aligned} \quad (15)$$

The fields $\tilde{\mathbf{R}}$, $\tilde{\mathbf{L}}$, $\tilde{\mathbf{U}}$, $\tilde{\mathbf{f}}$ and $\tilde{\mathbf{T}}$ are by construction divergence-free. The vector potential $\boldsymbol{\psi}$ of \mathbf{u} satisfies

$$\nabla \times \boldsymbol{\psi} = \mathbf{u}, \quad \nabla \cdot \boldsymbol{\psi} = 0. \quad (16)$$

For various purposes, it is reasonable to consider $\partial \bar{B}_i / \partial x_j$ as constant. Then we have

$$M(\mathbf{x}) = 2 \frac{\partial \bar{B}_i}{\partial x_j} \frac{\partial \tilde{b}_j(\mathbf{x})}{\partial x_i}, \quad \tilde{b}_j(\mathbf{x}) = \frac{1}{4\pi} \int_{\infty} \frac{b_j(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x'. \quad (17)$$

We note that $\nabla \cdot \tilde{\mathbf{b}} = 0$ and $\nabla^2 \tilde{\mathbf{b}} = -\mathbf{b}$. Analogous relations apply to V if $\partial \bar{U}_i / \partial x_j$ is constant. All quantities that occur in (15) to (17) may depend also on time, but for the sake of simplicity we have dropped the argument t everywhere.

3. The traditional approach (approach (i)) to the mean electromotive force \mathcal{E}

In view of a later comparison with τ -approaches, we repeat here the basic ideas of the traditional approach to the mean electromotive force \mathcal{E} and summarize some general results (see also, e.g., Krause and Rädler 1980 or Rädler 2000). We recall that many applications have been made in the astrophysical context (e.g., Rüdiger and Kitchatinov (1993), Kitchatinov *et al.* (1994)). For the sake of simplicity we exclude any mean motion of the fluid, that is, $\bar{\mathbf{U}} = \mathbf{0}$. If \mathbf{u} is given, \mathbf{b} is determined by equation (4) and proper initial and boundary conditions. This equation is inhomogeneous in \mathbf{b} due to the term with $\bar{\mathbf{B}}$. Any solution can be considered as a superposition of a solution of the homogeneous equation, which is independent of $\bar{\mathbf{B}}$, and a solution of the full equation which is linear and homogeneous in $\bar{\mathbf{B}}$. If there is a non-decaying solution of the homogeneous equation the mean electromotive force \mathcal{E} may have a non-decaying part, say $\mathcal{E}^{(0)}$, independent of $\bar{\mathbf{B}}$ (see, e.g., Rädler 1976, 2000). If $\nabla \times \mathcal{E}^{(0)} = \mathbf{0}$, such solutions correspond to small-scale dynamos. Until further notice we assume however that \mathbf{b} decays to zero as soon as $\bar{\mathbf{B}}$ is equal to zero. In this case, which will be discussed in section 3.1, we speak of “purely hydrodynamic background turbulence”. Later, in section 3.2, we will admit a non-decaying \mathbf{b} even if $\bar{\mathbf{B}}$ is equal to zero and then speak of “magneto-hydrodynamic background turbulence”. We use the word “turbulence” here in a wide sense. If not specified otherwise, \mathbf{u} and \mathbf{b} are considered as fields with arbitrary space and time dependencies but, of course, with zero averages.

3.1. Purely hydrodynamic background turbulence

3.1.1. Second-order correlation approximation. We start here with the induction equation (4) for the magnetic fluctuations \mathbf{b} . As already mentioned we exclude any mean motion, that is, put $\bar{\mathbf{U}} = \mathbf{0}$. We use the second-order correlation approximation, here defined by neglecting the term \mathbf{G} in equation (4). Then this equation takes the simple form

$$\partial_t \mathbf{b} - \eta \nabla^2 \mathbf{b} = \nabla \times (\mathbf{u} \times \bar{\mathbf{B}}), \quad \nabla \cdot \mathbf{b} = 0. \quad (18)$$

Until further notice a compressible fluid, that is, a non-zero $\nabla \cdot \mathbf{u}$ is admitted. We assume that equation (18) applies in all infinite space. Then its general solution can be written as

$$b_k(\mathbf{x}, t) = \int_{\infty} G^{(n)}(\mathbf{x} - \mathbf{x}', t - t_0) b_k(\mathbf{x}', t_0) d^3 x' + \epsilon_{klm} \epsilon_{mnp} \int_{\infty} \int_{t_0}^t G^{(n)}(\mathbf{x} - \mathbf{x}', t - t') \frac{\partial}{\partial x'_l} (u_n(\mathbf{x}', t') \bar{B}_p(\mathbf{x}', t')) d^3 x' dt', \quad (19)$$

where $G^{(n)}$ is a Green's function,

$$G^{(n)}(\mathbf{x}, t) = G^{(n)}(x, t) = (4\pi\eta t)^{-3/2} \exp(-x^2/4\eta t), \quad (20)$$

and $t = t_0$ means an initial time. We note that here, as a consequence of the neglect of \mathbf{G} , solutions \mathbf{b} which do not decay for $\bar{\mathbf{B}} = \mathbf{0}$ are automatically excluded.

For the calculation of \mathcal{E} we start from

$$\mathcal{E}_i(\mathbf{x}, t) = \epsilon_{ijk} \langle u_j(\mathbf{x}, t) b_k(\mathbf{x}, t) \rangle, \quad (21)$$

consider $t - t_0$ as sufficiently large so that $\langle u_j(\mathbf{x}, t) b_k(\mathbf{x}', t_0) \rangle = 0$ and let, for simplicity only, finally $t_0 \rightarrow -\infty$. In this way we find

$$\mathcal{E}_i(\mathbf{x}, t) = (\epsilon_{ijn} \delta_{lp} - \epsilon_{ijp} \delta_{ln}) \int_{\infty} \int_0^{\infty} \frac{\partial G^{(n)}(\xi, \tau) \xi_l}{\partial \xi} \frac{\xi_l}{\xi} Q_{jn}(\mathbf{x}, t; -\xi, -\tau) \bar{B}_p(\mathbf{x} - \xi, t - \tau) d^3 \xi d\tau \quad (22)$$

with the second-rank velocity correlation tensor Q_{ij} defined by

$$Q_{ij}(\mathbf{x}, t; \xi, \tau) = \langle u_i(\mathbf{x}, t) u_j(\mathbf{x} + \xi, t + \tau) \rangle. \quad (23)$$

We further assume sufficiently weak variations of $\bar{\mathbf{B}}$ in \mathbf{x} and t , and put under the integral in the sense of a truncated Taylor expansion

$$\bar{B}_p(\mathbf{x} - \xi, t - \tau) = \bar{B}_p(\mathbf{x}, t) - \xi_q \frac{\partial \bar{B}_p(\mathbf{x}, t)}{\partial x_q}, \quad (24)$$

ignoring all other terms, that is, those with $\tau \partial \bar{B}_p / \partial t$, $\xi_q \xi_r \partial^2 \bar{B}_p / \partial x_q \partial x_r$, \dots . In this way we find relation (6) with $\mathcal{E}_i^{(0)} = 0$, that is $\mathcal{E}_i = a_{ip} \bar{B}_p + b_{ipq} \partial \bar{B}_p / \partial x_q$, and

$$\begin{aligned} a_{ip} &= -(\epsilon_{ijn} \delta_{lp} - \epsilon_{ijp} \delta_{ln}) \int_{\infty} \int_0^{\infty} \frac{\partial G^{(n)}(\xi, \tau) \xi_l}{\partial \xi} \frac{\xi_l}{\xi} Q_{jn}(\mathbf{x}, t; \xi, -\tau) d^3 \xi d\tau \\ &= (\epsilon_{ijn} \delta_{lp} - \epsilon_{ijp} \delta_{ln}) \int_{\infty} \int_0^{\infty} G^{(n)}(\xi, \tau) \frac{\partial Q_{jn}(\mathbf{x}, t; \xi, -\tau)}{\partial \xi_l} d^3 \xi d\tau \\ b_{ipq} &= -(\epsilon_{ijn} \delta_{lp} - \epsilon_{ijp} \delta_{ln}) \int_{\infty} \int_0^{\infty} \frac{\partial G^{(n)}(\xi, \tau) \xi_l \xi_q}{\partial \xi} \frac{\xi_l \xi_q}{\xi} Q_{jn}(\mathbf{x}, t; \xi, -\tau) d^3 \xi d\tau. \end{aligned} \quad (25)$$

3.1.2. Range of validity. In order to formulate conditions under which the above results apply we introduce the magnetic Reynolds number Rm and the Strouhal number St ,

$$Rm = u_c \lambda_c / \eta, \quad St = u_c \tau_c / \lambda_c. \quad (26)$$

Here u_c means a characteristic value of the fluctuating velocity \mathbf{u} , and λ_c and τ_c are a characteristic length and a characteristic time of the variations of \mathbf{u} , respectively. If \mathbf{u} describes a fully developed turbulence, λ_c and τ_c can be interpreted as correlation length and correlation time. We further define the dimensionless parameter q by

$$q = \lambda_c^2 / \eta \tau_c. \quad (27)$$

It gives the ratio of the magnetic diffusion time λ_c^2 / η for a fluid element whose size is characterized by λ_c to the time τ_c . We have then $Rm / St = q$. If, e.g., St is fixed, the limit $Rm \rightarrow 0$ coincides with $q \rightarrow 0$, and $Rm \rightarrow \infty$ with $q \rightarrow \infty$. Sometimes the limits $q \rightarrow \infty$ and $q \rightarrow 0$ are denoted by “high-conductivity limit” and “low-conductivity limit”, respectively. This notation, introduced with the idea of varying η for fixed finite values of λ_c and τ_c , might however be misleading. If, e.g., λ_c and η are fixed, $q \rightarrow \infty$ or $q \rightarrow 0$ correspond to $\tau_c \rightarrow 0$ or $\tau_c \rightarrow \infty$, that is, very quick or very slow variation of \mathbf{u} in time, respectively.

The second-order correlation approximation applies as soon as the neglect of \mathbf{G} in (4) is justified. A sufficient condition for that is $|\mathbf{b}| \ll |\overline{\mathbf{B}}|$. If $q \lesssim 1$ this is equivalent to $Rm \ll 1$, if $q \gtrsim 1$ to $St \ll 1$. We may sum this up by saying that a sufficient condition reads

$$\min(Rm, St) \ll 1 \quad (28)$$

(see also, e.g., Krause and Rädler 1971, 1980, or Rädler 2000). Incidentally, determinations of a_{ij} and b_{ijk} have been done within the framework of numerical simulations of magnetoconvection or dynamos without using the second-order correlation approximation. It turned out that, e.g., for $q \lesssim 1$ there is a rather good agreement of the results for these quantities with those obtained in the second-order correlation approximation not only for $Rm \ll 1$ but for values of Rm up to about 10 (Schrunner *et al.* 2005, 2006, 2007).

3.1.3. Limiting cases concerning q . Let us specify the above results for a_{ip} and b_{ipq} to the limiting cases with respect to q . As shown in appendix A, in the limit $q \rightarrow \infty$ the relations (25) for a_{ip} and b_{ipq} turn into

$$\begin{aligned} a_{ip} &= (\epsilon_{ijn} \delta_{lp} - \epsilon_{ijp} \delta_{ln}) \int_0^\infty \left\langle u_j(\mathbf{x}, t) \frac{\partial u_n(\mathbf{x}, t - \tau)}{\partial x_l} \right\rangle d\tau \\ &= \epsilon_{ijn} \left(\int_0^\infty \left\langle u_j(\mathbf{x}, t) \frac{\partial u_n(\mathbf{x}, t - \tau)}{\partial x_p} \right\rangle d\tau - \delta_{np} \int_0^\infty \langle u_j(\mathbf{x}, t) (\nabla \cdot \mathbf{u}(\mathbf{x}, t - \tau)) \rangle d\tau \right) \\ b_{ipq} &= -\epsilon_{ijp} \int_0^\infty \langle u_j(\mathbf{x}, t) u_q(\mathbf{x}, t - \tau) \rangle d\tau. \end{aligned} \quad (29)$$

A contribution to b_{ipq} containing δ_{pq} , which is because of $\nabla \cdot \overline{\mathbf{B}} = 0$ meaningless for \mathcal{E} , has been omitted.

In the limit $q \rightarrow 0$ we obtain

$$\begin{aligned}
 a_{ip} &= \frac{\epsilon_{ijn}\delta_{lp} - \epsilon_{ijp}\delta_{ln}}{4\pi\eta} \int_{\infty} \langle u_j(\mathbf{x}, t) u_n(\mathbf{x} + \boldsymbol{\xi}, t) \rangle \xi_l \frac{d^3\xi}{\xi^3} \\
 &= \frac{\epsilon_{ijn}}{4\pi\eta} \left(\int_{\infty} \left\langle u_j(\mathbf{x}, t) \frac{\partial u_n(\mathbf{x} + \boldsymbol{\xi}, t)}{\partial x_p} \right\rangle \frac{d^3\xi}{\xi} - \delta_{np} \int_{\infty} \langle u_j(\mathbf{x}, t) (\nabla \cdot \mathbf{u}(\mathbf{x} + \boldsymbol{\xi}, t)) \rangle \frac{d^2\xi}{\xi} \right) \quad (30) \\
 b_{ipq} &= \frac{1}{4\pi\eta} \int_{\infty} (\epsilon_{ijn}\xi_p - \epsilon_{ijp}\xi_n) \xi_q \langle u_j(\mathbf{x}, t) u_n(\mathbf{x} + \boldsymbol{\xi}, t) \rangle \frac{d^3\xi}{\xi^3}.
 \end{aligned}$$

We note that these results for the limits $q \rightarrow \infty$ and $q \rightarrow 0$ can also be derived from equation (18) with $\eta\nabla^2\mathbf{b}$ or $\partial_t\mathbf{b}$, respectively, cancelled (Krause and Rädler 1971, 1980). Interestingly enough, for the limit $q \rightarrow \infty$ then, without introducing (24) or similar assumptions, no contributions to \mathcal{E} with higher than first-order spatial derivatives of $\overline{\mathbf{B}}$ occur. This is in agreement with the fact that, when considering first the general result for \mathcal{E} with higher spatial derivatives (Rädler 1968a,b) and then proceeding to the limit $q \rightarrow \infty$, all the coefficients of higher than first-order spatial derivatives vanish. This finding might give some justification for considering no contributions to \mathcal{E} with higher than first-order spatial derivatives in astrophysical applications in which the limit $q \rightarrow \infty$ is a good approximation. There is, however, no general justification to ignore contributions to \mathcal{E} with time derivatives of $\overline{\mathbf{B}}$. By contrast, in the case $q \rightarrow 0$ contributions to \mathcal{E} with higher spatial derivatives of $\overline{\mathbf{B}}$ are well possible but no contributions with time derivatives of $\overline{\mathbf{B}}$.

3.1.4. Homogeneous isotropic turbulence. It is instructive to consider the special case of a homogeneous isotropic turbulence, for which \mathcal{E} takes the form (7), that is $\mathcal{E} = \alpha\overline{\mathbf{B}} - \beta\nabla \times \overline{\mathbf{B}}$. For α and β , which are independent of position, we find then from (25)

$$\begin{aligned}
 \alpha &= -\frac{1}{3} \int_{\infty} \int_0^{\infty} \frac{\partial G^{(n)}(\boldsymbol{\xi}, \tau)}{\partial \xi} \langle \mathbf{u}(\mathbf{x}, t) \times \mathbf{u}(\mathbf{x} + \boldsymbol{\xi}, t - \tau) \rangle \cdot \frac{\boldsymbol{\xi}}{\xi} d^3\xi d\tau \\
 &= -\frac{1}{3} \int_{\infty} \int_0^{\infty} G^{(n)}(\boldsymbol{\xi}, \tau) \langle \mathbf{u}(\mathbf{x}, t) \cdot (\nabla \times \mathbf{u}(\mathbf{x} + \boldsymbol{\xi}, t - \tau)) \rangle d^3\xi d\tau \quad (31) \\
 \beta &= -\frac{1}{9} \int_{\infty} \int_0^{\infty} \frac{\partial G^{(n)}(\boldsymbol{\xi}, \tau)}{\partial \xi} \langle \mathbf{u}(\mathbf{x}, t) \cdot \mathbf{u}(\mathbf{x} + \boldsymbol{\xi}, t - \tau) \rangle \xi d^3\xi d\tau.
 \end{aligned}$$

Due to the assumed isotropy of the turbulence we may replace $\langle \mathbf{u}(\mathbf{x}, t) \cdot \mathbf{u}(\mathbf{x} + \boldsymbol{\xi}, t - \tau) \rangle$ by $3\langle u_{\xi}(\mathbf{x}, t) u_{\xi}(\mathbf{x} + \boldsymbol{\xi}, t - \tau) \rangle$, where $u_{\xi} = (\mathbf{u} \cdot \boldsymbol{\xi})/\xi$. Incidentally, since $G^{(n)}$ and all averages $\langle \dots \rangle$ depend on $\boldsymbol{\xi}$ via ξ only, we may replace the integrals $\int_{\infty} \dots d^3\xi$ by $4\pi \int_0^{\infty} \dots \xi^2 d\xi$.

In the limit $q \rightarrow \infty$ we have

$$\begin{aligned}
 \alpha &= -\frac{1}{3} \int_0^{\infty} \langle \mathbf{u}(\mathbf{x}, t) \cdot (\nabla \times \mathbf{u}(\mathbf{x}, t - \tau)) \rangle d\tau \\
 \beta &= \frac{1}{3} \int_0^{\infty} \langle \mathbf{u}(\mathbf{x}, t) \cdot \mathbf{u}(\mathbf{x}, t - \tau) \rangle d\tau.
 \end{aligned} \quad (32)$$

This result is often written in the form

$$\alpha = -\frac{1}{3} \langle \mathbf{u} \cdot (\nabla \times \mathbf{u}) \rangle \tau^{(\alpha)}, \quad \beta = \frac{1}{3} \langle \mathbf{u}^2 \rangle \tau^{(\beta)}. \quad (33)$$

Here $\tau^{(\alpha)}$ and $\tau^{(\beta)}$ are just defined by equating the right-hand sides of (32) to those of (33), respectively. This is, of course, only possible under the reasonable assumption that $\langle \mathbf{u}(\mathbf{x}, t) \cdot (\nabla \times \mathbf{u}(\mathbf{x}, t - \tau)) \rangle$ at $\tau = 0$ is unequal to zero. It seems plausible to assume that both $\tau^{(\alpha)}$ and $\tau^{(\beta)}$ are of the order of the correlation time τ_c of the turbulent velocity field. There is, however, hardly a general reason for their exact equality.

Remarkably, as explained in appendix B, in the case of an incompressible fluid and statistically steady turbulence the right-hand sides of (32) can also be expressed by integrals containing $\langle \mathbf{u}(\mathbf{x}, t) \cdot (\nabla \times \mathbf{u}(\mathbf{x} + \boldsymbol{\xi}, t)) \rangle$ or $\langle \mathbf{u}(\mathbf{x}, t) \cdot \mathbf{u}(\mathbf{x} + \boldsymbol{\xi}, t) \rangle$, taken over all $\boldsymbol{\xi}$.

In the limit $q \rightarrow 0$ we obtain

$$\begin{aligned} \alpha &= \frac{1}{12\pi\eta} \int_{\infty} \langle (\mathbf{u}(\mathbf{x}, t) \times \mathbf{u}(\mathbf{x} + \boldsymbol{\xi}, t)) \cdot \boldsymbol{\xi} \rangle \frac{d^3\xi}{\xi^3} = -\frac{1}{12\pi\eta} \int_{\infty} \langle \mathbf{u}(\mathbf{x}, t) \cdot (\nabla \times \mathbf{u}(\mathbf{x} + \boldsymbol{\xi}, t)) \rangle \frac{d^3\xi}{\xi} \\ \beta &= \frac{1}{36\pi\eta} \int_{\infty} \langle \mathbf{u}(\mathbf{x}, t) \cdot \mathbf{u}(\mathbf{x} + \boldsymbol{\xi}, t) \rangle \frac{d^3\xi}{\xi}. \end{aligned} \quad (34)$$

Again the integrals $\int_{\infty} \dots d^3\xi$ may be replaced by $4\pi \int_0^{\infty} \dots \xi^2 d\xi$.

Incidentally, the results (34) take a simple form if we represent \mathbf{u} by $\mathbf{u} = \nabla \times \boldsymbol{\psi} - \nabla\phi$ with the vector potential $\boldsymbol{\psi}$ and a scalar potential ϕ . They read then

$$\alpha = -\frac{1}{3\eta} \langle \mathbf{u} \cdot \boldsymbol{\psi} \rangle = -\frac{1}{3\eta} \langle \boldsymbol{\psi} \cdot (\nabla \times \boldsymbol{\psi}) \rangle, \quad \beta = \frac{1}{3\eta} (\langle \boldsymbol{\psi}^2 \rangle - \langle \phi^2 \rangle) \quad (35)$$

(see, e.g., Krause and Rädler 1971, 1980, but note that there is a sign error in Krause and Rädler 1971).

3.1.5. Comments. In the above calculation of the electromotive force \mathcal{E} no other restrictions concerning the fluctuating fluid velocity \mathbf{u} have been used than some ‘‘smallness’’ which ensures the applicability of the second-order correlation approximation.

We note that in this approximation if, as so far done, a purely hydrodynamic turbulence is considered and any influence of the initial \mathbf{b} is ignored, \mathcal{E} does not contain a part $\mathcal{E}^{(0)}$.

We may of course specify the velocity \mathbf{u} or its correlation tensor Q_{ij} such that they correspond to a turbulence on a rotating body, that is, subject to a Coriolis force. The dependence of \mathbf{u} and Q_{ij} on the angular velocity $\boldsymbol{\Omega}$ responsible for the Coriolis force has to be calculated on the basis of the momentum balance (14). Likewise we may consider a turbulence under the influence of a given mean magnetic field, that is, under the action of the Lorentz force, and specify \mathbf{u} , again on the basis of the momentum balance, to be a function of $\bar{\mathbf{B}}$ and its derivatives. In that sense the above

derivations may serve also as a starting point for investigations of the quenching of the mean-field induction effects by the mean magnetic field.

As for higher than second-order correlation approximations we refer to section 5.1.

3.2. Magneto hydrodynamic background turbulence

3.2.1. Second-order correlation approximation. Let us relax the assumption that \mathbf{b} decays to zero if $\bar{\mathbf{B}}$ vanishes, that is, admit a magneto hydrodynamic background turbulence. We assume the fluid to be incompressible. In addition to the induction equation (4) we use now from the very beginning also the momentum balance in the form (14). For the sake of simplicity we restrict our attention to a non-rotating fluid, that is $\boldsymbol{\Omega} = \mathbf{0}$, and exclude again any mean motion, $\bar{\mathbf{U}} = \mathbf{0}$. Then \mathbf{u} and \mathbf{b} are governed by

$$\begin{aligned} \partial_t \mathbf{u} &= \nu \nabla^2 \mathbf{u} + \frac{1}{\mu_0} ((\bar{\mathbf{B}} \cdot \nabla) \mathbf{b} + (\mathbf{b} \cdot \nabla) \bar{\mathbf{B}} + \nabla M) + \tilde{\mathbf{f}} + \tilde{\mathbf{T}}, \quad \nabla \cdot \mathbf{u} = 0 \\ \partial_t \mathbf{b} &= \eta \nabla^2 \mathbf{b} + (\bar{\mathbf{B}} \cdot \nabla) \mathbf{u} - (\mathbf{u} \cdot \nabla) \bar{\mathbf{B}} + \mathbf{G}, \quad \nabla \cdot \mathbf{b} = 0 \end{aligned} \quad (36)$$

with M , $\tilde{\mathbf{f}}$ and $\tilde{\mathbf{T}}$ as defined in (15).

We expand \mathbf{u} and \mathbf{b} with respect to $\bar{\mathbf{B}}$,

$$\mathbf{u} = \mathbf{u}^{(0)} + \mathbf{u}^{(1)} + \dots, \quad \mathbf{b} = \mathbf{b}^{(0)} + \mathbf{b}^{(1)} + \dots, \quad (37)$$

where $\mathbf{u}^{(0)}$ and $\mathbf{b}^{(0)}$ are independent of $\bar{\mathbf{B}}$, further $\mathbf{u}^{(1)}$ and $\mathbf{b}^{(1)}$ of first order in $\bar{\mathbf{B}}$, and ... stands for terms of higher order in $\bar{\mathbf{B}}$ (see also Blackman and Field 1999, or Field *et al.* 1999). Then we have

$$\begin{aligned} \boldsymbol{\mathcal{E}} &= \boldsymbol{\mathcal{E}}^{(0)} + \boldsymbol{\mathcal{E}}^{(1)} + \dots, \quad \boldsymbol{\mathcal{E}}^{(1)} = \boldsymbol{\mathcal{E}}^{(01)} + \boldsymbol{\mathcal{E}}^{(10)}, \dots \\ \boldsymbol{\mathcal{E}}^{(0)} &= \langle \mathbf{u}^{(0)} \times \mathbf{b}^{(0)} \rangle, \quad \boldsymbol{\mathcal{E}}^{(01)} = \langle \mathbf{u}^{(0)} \times \mathbf{b}^{(1)} \rangle, \quad \boldsymbol{\mathcal{E}}^{(10)} = \langle \mathbf{u}^{(1)} \times \mathbf{b}^{(0)} \rangle, \dots \end{aligned} \quad (38)$$

We split now the equations (36) in the usual way into equations for $\mathbf{u}^{(0)}$ and $\mathbf{b}^{(0)}$, which do not contain $\mathbf{u}^{(1)}$ and $\mathbf{b}^{(1)}$, and equations for $\mathbf{u}^{(1)}$ and $\mathbf{b}^{(1)}$, which contain of course also $\mathbf{u}^{(0)}$ and $\mathbf{b}^{(0)}$. We consider those for $\mathbf{u}^{(0)}$ and $\mathbf{b}^{(0)}$ as being solved, that is, $\mathbf{u}^{(0)}$ and $\mathbf{b}^{(0)}$ as given. When deriving the equations for $\mathbf{u}^{(1)}$ and $\mathbf{b}^{(1)}$ we assume that \mathbf{f} is independent of $\bar{\mathbf{B}}$. We further use the second-order approximation, in this context understood as the neglect of $\mathbf{G}^{(1)}$ and $\tilde{\mathbf{T}}^{(1)}$ in these equations. Here $\mathbf{G}^{(1)}$ is defined by $\mathbf{G}^{(1)} = \nabla \times ((\mathbf{u}^{(0)} \times \mathbf{b}^{(1)})' + (\mathbf{u}^{(1)} \times \mathbf{b}^{(0)}))$ and $\tilde{\mathbf{T}}^{(1)}$ is a quantity derived in the sense of (13) and (15) from the analogously defined $\mathbf{T}^{(1)}$. In this way we arrive at

$$\begin{aligned} \partial_t \mathbf{u}^{(1)} - \nu \nabla^2 \mathbf{u}^{(1)} &= \frac{1}{\mu_0} ((\bar{\mathbf{B}} \cdot \nabla) \mathbf{b}^{(0)} + (\mathbf{b}^{(0)} \cdot \nabla) \bar{\mathbf{B}} + \nabla M^{(1)}), \quad \nabla \cdot \mathbf{u}^{(1)} = 0 \\ \partial_t \mathbf{b}^{(1)} - \eta \nabla^2 \mathbf{b}^{(1)} &= (\bar{\mathbf{B}} \cdot \nabla) \mathbf{u}^{(0)} - (\mathbf{u}^{(0)} \cdot \nabla) \bar{\mathbf{B}}, \quad \nabla \cdot \mathbf{b}^{(1)} = 0. \end{aligned} \quad (39)$$

$M^{(1)}$ is defined as M in (15) but with $\mathbf{b}^{(0)}$ instead of \mathbf{b} .

We recall here the derivations of section 3.1.1. Starting from equation (18) we have there calculated \mathcal{E}_i in the form (22). With the second equation (39) we find in the same way

$$\mathcal{E}_i^{(01)}(\mathbf{x}, t) = (\epsilon_{ijn}\delta_{lp} - \epsilon_{ijp}\delta_{ln}) \int_{-\infty}^{\infty} \int_0^{\infty} \frac{\partial G^{(n)}(\xi, \tau) \xi_l}{\partial \xi} \frac{\xi_l}{\xi} Q_{jn}^{(0)}(\mathbf{x}, t; -\xi, -\tau) \bar{B}_p(\mathbf{x} - \xi, t - \tau) d^3\xi d\tau. \quad (40)$$

$Q_{ij}^{(0)}$ is in analogy to Q_{ij} defined by

$$Q_{ij}^{(0)}(\mathbf{x}, t; \xi, \tau) = \left\langle u_i^{(0)}(\mathbf{x}, t) u_j^{(0)}(\mathbf{x} + \xi, t + \tau) \right\rangle. \quad (41)$$

Because of $\nabla \cdot \mathbf{u}^{(0)} = 0$ we have $\partial Q_{ij}^{(0)}/\partial x_i - \partial Q_{ij}^{(0)}/\partial \xi_i = 0$ and $\partial Q_{ij}^{(0)}/\partial \xi_j = 0$.

The first equation (39) leads to

$$\begin{aligned} \mathcal{E}_i^{(10)}(\mathbf{x}, t) = & -\frac{\epsilon_{ijn}\delta_{lp} + \epsilon_{ijp}\delta_{ln}}{\mu\varrho} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{\partial G^{(v)}(\xi, \tau) \xi_l}{\partial \xi} \frac{\xi_l}{\xi} R_{jn}^{(0)}(\mathbf{x}, t; -\xi, -\tau) \bar{B}_p(\mathbf{x} - \xi, t - \tau) d^3\xi d\tau \\ & - \frac{2\epsilon_{ijl}}{\mu\varrho} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{\partial G^{(v)}(\xi, \tau) \xi_l}{\partial \xi} \frac{\xi_l}{\xi} N_j^{(0)}(\mathbf{x}, t; -\xi, -\tau) d^3\xi d\tau. \end{aligned} \quad (42)$$

$G^{(v)}(\xi, \tau)$ is defined like $G^{(n)}(\xi, \tau)$ in (20), but with η replaced by ν . Further $R_{ij}^{(0)}$ is analogously to $Q_{ij}^{(0)}$ defined by

$$R_{ij}^{(0)}(\mathbf{x}, t; \xi, \tau) = \left\langle b_i^{(0)}(\mathbf{x}, t) b_j^{(0)}(\mathbf{x} + \xi, t + \tau) \right\rangle \quad (43)$$

and $N_j^{(0)}$ by

$$N_j^{(0)}(\mathbf{x}, t; \xi, \tau) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\partial R_{jn}^{(0)}(\mathbf{x}, t; \xi', \tau)}{\partial \xi'_p} \frac{\partial \bar{B}_p(\mathbf{x} + \xi', t + \tau)}{\partial \xi'_n} \frac{d^3\xi'}{|\xi - \xi'|}. \quad (44)$$

Because of $\nabla \cdot \mathbf{b}^{(0)} = 0$ we have $\partial R_{ij}^{(0)}/\partial x_i - \partial R_{ij}^{(0)}/\partial \xi_i = 0$ and $\partial R_{ij}^{(0)}/\partial \xi_j = 0$.

We restrict our attention now to the limit of small $\bar{\mathbf{B}}$. As in section 3.1.1 we assume in addition sufficiently weak variations of $\bar{\mathbf{B}}$ in \mathbf{x} and t and introduce the truncated Taylor expansion (24) into the integrands of (40), (42) and (44). Considering (38) and ignoring contributions to \mathcal{E} of higher than first order in $\bar{\mathbf{B}}$ we find then again (6), that is $\mathcal{E}_i = \mathcal{E}_i^{(0)} + a_{ij} \bar{B}_j + b_{ijk} \partial \bar{B}_j / \partial x_k$. Here we have $\mathcal{E}^{(0)} = \langle \mathbf{u}^{(0)} \times \mathbf{b}^{(0)} \rangle$, and this may in general well be unequal to zero. We put

$$a_{ij} = a_{ij}^{(u)} + a_{ij}^{(b)}, \quad b_{ijk} = b_{ijk}^{(u)} + b_{ijk}^{(b)}, \quad (45)$$

with $a_{ij}^{(u)}$ and $b_{ijk}^{(u)}$ determined by $\mathbf{u}^{(0)}$, and $a_{ij}^{(b)}$ and $b_{ijk}^{(b)}$ by $\mathbf{b}^{(0)}$. Then we have

$$\begin{aligned}
 a_{ip}^{(u)} &= \epsilon_{ijk} \int_{-\infty}^{\infty} \int_0^{\infty} G^{(n)}(\xi, \tau) \frac{\partial Q_{jk}^{(0)}(\mathbf{x}, t; \xi, -\tau)}{\partial \xi_p} d^3 \xi d\tau \\
 &= -\epsilon_{ijk} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{\partial G^{(n)}(\xi, \tau)}{\partial \xi} \frac{\xi_p}{\xi} Q_{jk}^{(0)}(\mathbf{x}, t; \xi, -\tau) d^3 \xi d\tau \\
 a_{ip}^{(b)} &= -\frac{\epsilon_{ijk}}{\mu Q} \int_{-\infty}^{\infty} \int_0^{\infty} G^{(v)}(\xi, \tau) \frac{\partial R_{jk}^{(0)}(\mathbf{x}, t; \xi, -\tau)}{\partial \xi_p} d^3 \xi d\tau \\
 &= \frac{\epsilon_{ijk}}{\mu Q} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{\partial G^{(v)}(\xi, \tau)}{\partial \xi} \frac{\xi_p}{\xi} R_{jk}^{(0)}(\mathbf{x}, t; \xi, -\tau) d^3 \xi d\tau \\
 b_{ipq}^{(u)} &= -\epsilon_{ijn} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{\partial G^{(n)}(\xi, \tau)}{\partial \xi} \frac{\xi_p \xi_q}{\xi} Q_{jn}^{(0)}(\mathbf{x}, t; \xi, -\tau) d^3 \xi d\tau \\
 &\quad - \epsilon_{ijp} \int_{-\infty}^{\infty} \int_0^{\infty} G^{(n)}(\xi, \tau) Q_{jq}^{(0)}(\mathbf{x}, t; \xi, -\tau) d^3 \xi d\tau \\
 b_{ipq}^{(b)} &= \frac{\epsilon_{ijn}}{\mu Q} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{\partial G^{(v)}(\xi, \tau)}{\partial \xi} \frac{\xi_p \xi_q}{\xi} R_{jn}^{(0)}(\mathbf{x}, t; \xi, -\tau) d^3 \xi d\tau \\
 &\quad - \frac{1}{\mu Q} \int_{-\infty}^{\infty} \int_0^{\infty} G^{(v)}(\xi, \tau) \left(\epsilon_{ijp} - 2 \epsilon_{ijl} \frac{\partial^2}{\partial \xi_l \partial \xi_p} \Delta_{\xi}^{-1} \right) R_{jq}^{(0)}(\mathbf{x}, t; \xi, -\tau) d^3 \xi d\tau,
 \end{aligned} \tag{46}$$

where

$$\Delta_{\xi}^{-1} F(\mathbf{x}, t; \xi, \tau) = -\frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{F(\mathbf{x}, t; \xi', \tau)}{|\xi - \xi'|} d^3 \xi'. \tag{47}$$

By the way, the above relations for $a_{ip}^{(u)}$ and $b_{ipq}^{(u)}$ can also be concluded from (25) by replacing a_{ip} , b_{ipq} and Q_{jn} with $a_{ip}^{(u)}$, $b_{ipq}^{(u)}$ and $Q_{jn}^{(0)}$, respectively, and considering $\partial Q_{ij}^{(0)} / \partial \xi_j = 0$.

3.2.2. Range of validity. In addition to the dimensionless parameters Rm , St and q introduced with (26) and (27) we define the hydrodynamic Reynolds number Re and a parameter p by

$$Re = u_c \lambda_c / \nu, \quad p = \lambda_c^2 / \nu \tau_c. \tag{48}$$

We might call the cases $p \rightarrow \infty$ and $p \rightarrow 0$ “low-viscosity limit” and “high-viscosity limit”. However, the remarks made on the notations “high-conductivity limit” and “low-conductivity limit” made under (27) apply analogously.

Modifying the reasoning which lead us to the condition (28) properly we find that a sufficient condition for the validity of our above results for magnetohydrodynamic background turbulence is given by

$$\min(Rm, St) \ll 1, \quad \min(Re, St) \ll 1. \tag{49}$$

3.2.3. A limiting case. We note that $a_{ij}^{(u)}$ and $b_{ijk}^{(u)}$ depend only on q and not on p , and $a_{ij}^{(b)}$ and $b_{ijk}^{(b)}$ only on p and not on q . The relations for $a_{ij}^{(u)}$ and $b_{ijk}^{(u)}$ in the limits $q \rightarrow \infty$ and $q \rightarrow 0$ agree with those for a_{ij} and b_{ijk} given with (29) and (30), specified to $\nabla \cdot \mathbf{u} = 0$ and with \mathbf{u} replaced by $\mathbf{u}^{(0)}$. For $a_{ij}^{(b)}$ and $b_{ijk}^{(b)}$ we find on the way described in appendix A in the limit $p \rightarrow \infty$

$$\begin{aligned} a_{ip}^{(b)} &= -\frac{\epsilon_{ijk}}{\mu_Q} \int_0^\infty \left\langle b_j^{(0)}(\mathbf{x}, t) \frac{\partial b_k^{(0)}(\mathbf{x}, t - \tau)}{\partial x_p} \right\rangle d\tau \\ b_{ipq}^{(b)} &= -\frac{\epsilon_{ijp}}{\mu_Q} \int_0^\infty \left\langle b_j^{(0)}(\mathbf{x}, t) b_q^{(0)}(\mathbf{x}, t - \tau) \right\rangle d\tau \\ &\quad - \frac{\epsilon_{ijl}}{2\pi\mu_Q} \int_\infty \int_0^\infty \frac{\partial^2}{\partial \xi_j \partial \xi_p} \left(\frac{1}{\xi} \right) \left\langle b_j^{(0)}(\mathbf{x}, t) b_q^{(0)}(\mathbf{x} + \boldsymbol{\xi}, t - \tau) \right\rangle d^3 \xi d\tau. \end{aligned} \quad (50)$$

A contribution to $b_{ipq}^{(b)}$ containing δ_{pq} has been omitted. In the limit $p \rightarrow 0$ we have

$$\begin{aligned} a_{ip}^{(b)} &= -\frac{\epsilon_{ijk}}{4\pi\mu_Q v} \int_\infty \left\langle b_j^{(0)}(\mathbf{x}, t) b_k^{(0)}(\mathbf{x} + \boldsymbol{\xi}, t) \right\rangle \xi_p \frac{d^3 \xi}{\xi^3} \\ &= -\frac{\epsilon_{ijk}}{4\pi\mu_Q v} \int_\infty \left\langle b_j^{(0)}(\mathbf{x}, t) \frac{\partial b_k^{(0)}(\mathbf{x} + \boldsymbol{\xi}, t)}{\partial x_p} \right\rangle \frac{d^3 \xi}{\xi} \\ b_{ipq}^{(b)} &= -\frac{\epsilon_{ijm}}{4\pi\mu_Q v} \int_\infty \left(\delta_{mp} \delta_{nq} + \delta_{mn} \frac{\xi_p \xi_q}{\xi^2} \right) \left\langle b_j^{(0)}(\mathbf{x}, t) b_n^{(0)}(\mathbf{x} + \boldsymbol{\xi}, t) \right\rangle \frac{d^3 \xi}{\xi} \\ &\quad - \frac{\epsilon_{ijl}}{8\pi^2 \mu_Q v} \int_\infty \int_\infty \frac{\partial^2}{\partial \xi_j' \partial \xi_p'} \left(\frac{1}{|\boldsymbol{\xi} - \boldsymbol{\xi}'|} \right) \left\langle b_j^{(0)}(\mathbf{x}, t) b_q^{(0)}(\mathbf{x} + \boldsymbol{\xi}', t) \right\rangle d^3 \xi' \frac{d^3 \xi}{\xi}. \end{aligned} \quad (51)$$

3.2.4. Homogeneous isotropic turbulence. Let us now again consider the special case of homogeneous isotropic magnetohydrodynamic turbulence. Then \mathcal{E} takes again the form (7), that is $\mathcal{E} = \alpha \mathbf{B} - \beta \nabla \times \mathbf{B}$. On the basis of (45) and (46) we find

$$\alpha = \alpha^{(u)} - \alpha^{(b)}, \quad \beta = \beta^{(u)} \quad (52)$$

with

$$\begin{aligned} \alpha^{(u)} &= -\frac{1}{3} \int_\infty \int_0^\infty \frac{\partial G^{(n)}(\boldsymbol{\xi}, \tau)}{\partial \xi} \left\langle \mathbf{u}^{(0)}(\mathbf{x}, t) \times \mathbf{u}^{(0)}(\mathbf{x} + \boldsymbol{\xi}, t - \tau) \right\rangle \cdot \frac{\boldsymbol{\xi}}{\xi} d^3 \xi d\tau \\ &= -\frac{1}{3} \int_\infty \int_0^\infty G^{(n)}(\boldsymbol{\xi}, \tau) \left\langle \mathbf{u}^{(0)}(\mathbf{x}, t) \cdot (\nabla \times \mathbf{u}^{(0)}(\mathbf{x} + \boldsymbol{\xi}, t - \tau)) \right\rangle d^3 \xi d\tau \\ \alpha^{(b)} &= -\frac{1}{3\mu_Q} \int_\infty \int_0^\infty \frac{\partial G^{(v)}(\boldsymbol{\xi}, \tau)}{\partial \xi} \left\langle \mathbf{b}^{(0)}(\mathbf{x}, t) \times \mathbf{b}^{(0)}(\mathbf{x} + \boldsymbol{\xi}, t - \tau) \right\rangle \cdot \frac{\boldsymbol{\xi}}{\xi} d^3 \xi d\tau \\ &= -\frac{1}{3\mu_Q} \int_\infty \int_0^\infty G^{(v)}(\boldsymbol{\xi}, \tau) \left\langle \mathbf{b}^{(0)}(\mathbf{x}, t) \cdot (\nabla \times \mathbf{b}^{(0)}(\mathbf{x} + \boldsymbol{\xi}, t - \tau)) \right\rangle d^3 \xi d\tau \\ \beta^{(u)} &= \frac{1}{3} \int_\infty \int_0^\infty G^{(n)}(\boldsymbol{\xi}, \tau) \left\langle \mathbf{u}^{(0)}(\mathbf{x}, t) \cdot \mathbf{u}^{(0)}(\mathbf{x} + \boldsymbol{\xi}, t - \tau) \right\rangle d^3 \xi d\tau. \end{aligned} \quad (53)$$

By the reason explained in the context of (31) the integrals $\int_{\infty} \dots d^3\xi$ in (53) may be replaced by $4\pi \int_0^{\infty} \dots \xi^2 d\xi$. In addition to the “kinetic part” $\alpha^{(u)}$ of α there is a “magnetic part” $\alpha^{(b)}$, too. Interestingly enough there is no “magnetic part” of β . This agrees with a result by Vainshtein and Kichatinov (1983) obtained within an early version of the τ -approach applying to the limit $q \rightarrow \infty$.

In the case $q, p \rightarrow \infty$ we have

$$\begin{aligned} \alpha^{(u)} &= -\frac{1}{3} \int_0^{\infty} \langle \mathbf{u}^{(0)}(\mathbf{x}, t) \cdot (\nabla \times \mathbf{u}^{(0)}(\mathbf{x}, t - \tau)) \rangle d\tau \\ \alpha^{(b)} &= -\frac{1}{3\mu_0} \int_0^{\infty} \langle \mathbf{b}^{(0)}(\mathbf{x}, t) \cdot (\nabla \times \mathbf{b}^{(0)}(\mathbf{x}, t - \tau)) \rangle d\tau \\ \beta^{(u)} &= \frac{1}{3} \int_0^{\infty} \langle \mathbf{u}^{(0)}(\mathbf{x}, t) \cdot \mathbf{u}^{(0)}(\mathbf{x}, t - \tau) \rangle d\tau. \end{aligned} \tag{54}$$

In the sense discussed in the context of (33) we write this also in the form

$$\begin{aligned} \alpha^{(u)} &= -\frac{1}{3} \langle \mathbf{u}^{(0)} \cdot (\nabla \times \mathbf{u}^{(0)}) \rangle \tau^{(\alpha u)}, \quad \alpha^{(b)} = -\frac{1}{3\mu_0} \langle \mathbf{b}^{(0)} \cdot (\nabla \times \mathbf{b}^{(0)}) \rangle \tau^{(\alpha b)} \\ \beta^{(u)} &= \frac{1}{3} \langle \mathbf{u}^{(0)2} \rangle \tau^{(\beta u)} \end{aligned} \tag{55}$$

with times $\tau^{(\alpha u)}$, $\tau^{(\alpha b)}$ and $\tau^{(\beta u)}$ defined by equating the corresponding right-hand sides of (54) and (55).

3.2.5. Comments. In the results for purely hydrodynamic background turbulence presented in section 3.1 the velocity \mathbf{u} may well depend on $\overline{\mathbf{B}}$. If we ignore this dependence \mathbf{u} agrees with $\mathbf{u}^{(0)}$ introduced here. In this case the results of section 3.1 coincide with those obtained here if specified to $\mathbf{b}^{(0)} = \mathbf{0}$. We stress that in the case of purely hydrodynamic background turbulence, even if we assume that \mathbf{u} is subject to the influence of the Lorentz force and so depends on $\overline{\mathbf{B}}$, e.g., the relations (25), (29) and (30) for a_{ip} retain their validity and do not have to be modified by adding terms which explicitly depend on \mathbf{b} . Only if we replace there \mathbf{u} by $\mathbf{u}^{(0)}$ an additional term containing \mathbf{b} might occur.

Referring to Pouquet *et al.* (1976) often a relation for α as given by (52) and (55), but with $\mathbf{u}^{(0)}$ and $\mathbf{b}^{(0)}$ replaced by the real \mathbf{u} and \mathbf{b} , is taken to be valid. When applying this relation to purely hydrodynamic background turbulence, in contrast to our result (32) a “magnetic part” of α occurs. It is then claimed that this part occurs as a consequence of the mean magnetic field and hence it is discussed in the context of α -quenching. We note however that the interpretation of Pouquet *et al.* (in subsection “Phenomenology of the M-helicity effect” of section 3 of their paper) is based on the assumption of a (at least initially) purely magnetic background turbulence, see their equation (3.19). It has to be considered as a consequence of a given electromotive force rather than a kinetic driving force. Under these conditions, too, it is not surprising to see a term depending on the current helicity $\langle \mathbf{b} \cdot (\nabla \times \mathbf{b}) \rangle$ in the expression for α .

4. The τ -approaches (approaches (ii)) to the mean electromotive force \mathcal{E}

As mentioned above, several analytical approaches to the mean electromotive force \mathcal{E} going beyond the second-order correlation approximation have been proposed which make use of some modification of the τ -approximation of hydrodynamic turbulence theory (Orszag 1970). In contrast to approach (i), in these approaches, labelled by (ii) in the following, both the induction equation (4) and the momentum balance in the form (14) with the forcing term $\tilde{\mathbf{f}}$ and the non-linear terms \mathbf{G} and $\tilde{\mathbf{T}}$ are used from the very beginning. In this way also $\nabla \cdot \mathbf{u} = 0$ is introduced. Equations (4) and (14) are however not really solved. Instead, relations for \mathcal{E} are derived which contain, of course, contributions resulting from the mentioned quantities, that is, $\tilde{\mathbf{f}}$, \mathbf{G} and $\tilde{\mathbf{T}}$. For these contributions then some kind of τ -approximation is introduced. In this section we give a critical review of the approaches of type (ii). In particular we check to what extent their results satisfy the elementary requirement that they agree with results of approach (i) in the range of its validity.

For the sake of simplicity we exclude again any mean motion of the fluid, that is, $\bar{\mathbf{U}} = \mathbf{0}$. We further restrict our attention to the case of a non-rotating fluid, that is, $\bar{\boldsymbol{\Omega}} = \mathbf{0}$, and give only short comments on the case with $\bar{\boldsymbol{\Omega}} \neq \mathbf{0}$.

4.1. The simple τ -approach (approach (iia))

Let us first describe a simple approach to the mean electromotive force \mathcal{E} , which we call approach (iia) in the following. It defines a frame which allows us to explain and to discuss the approaches used by Vainshtein and Kichatinov (1983), Blackman and Field (2002), and Brandenburg and Subramanian (2005b).

4.1.1. Outline of approach (iia). We calculate first the time derivative of \mathcal{E} on the basis of

$$\partial_t \mathcal{E} = \langle \partial_t \mathbf{u} \times \mathbf{b} + \mathbf{u} \times \partial_t \mathbf{b} \rangle. \quad (56)$$

Using equations (4), further (14) with $\tilde{\mathbf{R}} = \tilde{\mathbf{U}} = \mathbf{0}$, and considering $\partial \bar{\mathbf{B}}_p / \partial x_q$ as constant so that (17) applies, we obtain

$$\partial_t \mathcal{E} = \mathbf{I} + \mathbf{Y} + \mathbf{Z}, \quad (57)$$

where

$$\begin{aligned} I_i &= \epsilon_{ijk} \left(\left\langle u_j \frac{\partial u_k}{\partial x_p} \right\rangle - \frac{1}{\mu_Q} \left\langle b_j \frac{\partial b_k}{\partial x_p} \right\rangle \right) \bar{B}_p - \epsilon_{ijk} \left(\langle u_j u_q \rangle \delta_{kp} - \frac{1}{\mu_Q} \left(\langle b_k b_q \rangle \delta_{jp} + 2 \left\langle \frac{\partial^2 \tilde{b}_q}{\partial x_j \partial x_p} b_k \right\rangle \right) \right) \frac{\partial \bar{B}_p}{\partial x_q} \\ Y_i &= \epsilon_{ijk} \langle v(\Delta u_j) b_k + \eta u_j \Delta b_k \rangle + \epsilon_{ijk} \langle \tilde{f}_j b_k \rangle \\ Z_i &= \epsilon_{ijk} \langle \tilde{T}_j b_k + u_j G_k \rangle. \end{aligned} \quad (58)$$

The term \mathbf{Y} depends on v and η . Even if these quantities vanish it remains non-zero as long as $\epsilon_{ijk} \langle \tilde{f}_j b_k \rangle$ does so. The term \mathbf{Z} considers the non-linear terms \mathbf{G} and $\tilde{\mathbf{T}}$ that occur in (4) and (14). It vanishes if these terms do so.

Let us now split \mathcal{E} , \mathbf{I} , \mathbf{Y} and \mathbf{Z} into parts $\mathcal{E}^{(0)}$, $\mathbf{I}^{(0)}$, $\mathbf{Y}^{(0)}$ and $\mathbf{Z}^{(0)}$, which are independent of $\bar{\mathbf{B}}$, and remaining parts $\mathcal{E}^{(B)}$, $\mathbf{I}^{(B)}$, $\mathbf{Y}^{(B)}$ and $\mathbf{Z}^{(B)}$. Clearly $\mathbf{I}^{(0)} = \mathbf{0}$, that is $\mathbf{I}^{(B)} = \mathbf{I}$. We do not deal with $\mathcal{E}^{(0)}$ and remark only that it is equal to zero for several simple cases. For $\mathcal{E}^{(B)}$ we have

$$\partial_t \mathcal{E}^{(B)} = \mathbf{I} + \mathbf{Y}^{(B)} + \mathbf{Z}^{(B)}. \tag{59}$$

We introduce now the assumptions

$$\mathbf{Y}^{(B)} = -\mathcal{E}^{(B)}/\tau_Y, \quad \mathbf{Z}^{(B)} = -\mathcal{E}^{(B)}/\tau_Z \tag{60}$$

with two times τ_Y and τ_Z , which are considered as non-negative. The second of these equations introduces some kind of τ -approximation, which reduces terms of third order in \mathbf{u} or \mathbf{b} to such of second order.

When accepting (60) we may rewrite (59) into

$$\partial_t \mathcal{E}^{(B)} + \frac{1}{\tau_*} \mathcal{E}^{(B)} = \mathbf{I}, \quad \frac{1}{\tau_*} = \frac{1}{\tau_Y} + \frac{1}{\tau_Z}. \tag{61}$$

This is equivalent to

$$\mathcal{E}^{(B)}(t) = \mathcal{E}^{(B)}(t_0) \exp\left(-\frac{t-t_0}{\tau_*}\right) + \int_0^{t-t_0} \mathbf{I}(t-t') \exp\left(-\frac{t'}{\tau_*}\right) dt', \tag{62}$$

where t_0 means some initial time. We let $t_0 \rightarrow -\infty$ and assume that \mathbf{I} does not markedly vary in time intervals with a length of some τ_* . In this way we find a result for \mathcal{E} which we can write in the form of (6), that is $\mathcal{E}_i = \mathcal{E}_i^{(0)} + a_{ip}\bar{B}_p + b_{ipq}\partial\bar{B}_p/\partial x_q$, with

$$\begin{aligned} a_{ip} &= \epsilon_{ijk} \left(\left\langle u_j \frac{\partial u_k}{\partial x_p} \right\rangle - \frac{1}{\mu\varrho} \left\langle b_j \frac{\partial b_k}{\partial x_p} \right\rangle \right) \tau_* \\ b_{ipq} &= -\epsilon_{ijk} \left(\left\langle u_j u_q \right\rangle \delta_{kp} - \frac{1}{\mu\varrho} \left(\left\langle b_k b_q \right\rangle \delta_{jp} + 2 \left\langle \frac{\partial^2 \tilde{b}_q}{\partial x_j \partial x_p} b_k \right\rangle \right) \right) \tau_*. \end{aligned} \tag{63}$$

Let us again consider the special case of a homogeneous isotropic turbulence. Then $\mathcal{E}^{(0)}$ has to be equal to zero and we have again $a_{ip} = \alpha \delta_{ip}$ and $b_{ipq} = \beta \epsilon_{ipq}$. Consequently \mathcal{E} takes the form (7), that is $\mathcal{E} = \alpha \bar{\mathbf{B}} - \beta \nabla \times \bar{\mathbf{B}}$. With (63) and considering $\nabla^2 \mathbf{b} = -\mathbf{b}$ we find

$$\alpha = -\frac{1}{3} \left(\langle \mathbf{u} \cdot (\nabla \times \mathbf{u}) \rangle - \frac{1}{\mu\varrho} \langle \mathbf{b} \cdot (\nabla \times \mathbf{b}) \rangle \right) \tau_*, \quad \beta = \frac{1}{3} \langle u^2 \rangle \tau_*. \tag{64}$$

There is a kinetic and a magnetic contribution to α but again only a kinetic one to β .

It is of some interest to consider also the case of a homogeneous turbulence deviating from isotropy only due to the presence of a homogeneous mean magnetic field $\bar{\mathbf{B}}$.

Then we have $a_{ip} = \alpha_1 \delta_{ip} + \alpha_2 e_{Bi} e_{Bp}$, where $\mathbf{e}_B = \bar{\mathbf{B}}/|\bar{\mathbf{B}}|$. This leads to $\mathcal{E} = (\alpha_1 + \alpha_2)\bar{\mathbf{B}}$ and $\alpha_1 + \alpha_2 = a_{ip} e_{Bi} e_{Bp}$. We may write again $\mathcal{E} = \alpha \bar{\mathbf{B}}$. However α has then no longer the form given by (64) but

$$\alpha = \left(\langle \mathbf{u} \times (\mathbf{e}_B \cdot \nabla) \mathbf{u} \rangle \cdot \mathbf{e}_B - \frac{1}{\mu Q} \langle \mathbf{b} \times (\mathbf{e}_B \cdot \nabla) \mathbf{b} \rangle \cdot \mathbf{e}_B \right) \tau_*. \quad (65)$$

It is in any case the dependence of \mathbf{u} and \mathbf{b} on $\bar{\mathbf{B}}$ which leads to a dependence of α on $|\bar{\mathbf{B}}|$. In addition a dependence of τ_* on $|\bar{\mathbf{B}}|$ cannot be ruled out.

In the limit of small $\bar{\mathbf{B}}$ the averages $\langle \dots \rangle$ in (63) and (64) turn into $\langle \dots \rangle^{(0)}$, what means that \mathbf{u} and \mathbf{b} are replaced by the corresponding quantities $\mathbf{u}^{(0)}$ and $\mathbf{b}^{(0)}$ for $\bar{\mathbf{B}} \rightarrow \mathbf{0}$. In view of this limit, we further remark that the average of the three values of α given by (65) with $\mathbf{e}_B = (1, 0, 0)$, $\mathbf{e}_B = (0, 1, 0)$ and $\mathbf{e}_B = (0, 0, 1)$ agrees just with α given by (64), again with \mathbf{u} and \mathbf{b} replaced by $\mathbf{u}^{(0)}$ and $\mathbf{b}^{(0)}$.

4.1.2. Comments. In the above derivation of the result (63) the assumptions (60) play a crucial role. If we simplify our reasoning by considering (61) from the very beginning for the steady case only, a very strange aspect of the way to a relation of type (6), $\mathcal{E}_i = \dots + a_{ip} \bar{B}_p + b_{ipq} \partial \bar{B}_p / \partial x_q$, becomes obvious. There is no original relation of this type, which then would be improved by the assumptions (60). Instead, this relation is just established by the assumptions (60). \mathcal{E} on the left-hand side of the relation corresponding to (6) originates only from the $\mathcal{E}^{(B)}$ in (60).

Nevertheless, at first glance, the assumptions (60) look plausible at least for simple cases. For example, for an isotropic background turbulence and a homogeneous steady mean magnetic field $\bar{\mathbf{B}}$ the quantities $\mathbf{Y}^{(B)}$, $\mathbf{Z}^{(B)}$ and $\mathcal{E}^{(B)}$ must, by symmetry reasons, be proportional to $\bar{\mathbf{B}}$. This leads immediately to relations like (60). The minus signs in these relations together with positive values τ_Y and τ_Z ensures that \mathcal{E} remains finite.

The assumptions (60) become however questionable in more complex cases. Consider, for example, again an isotropic background turbulence but admit an inhomogeneous steady field $\bar{\mathbf{B}}$. Clearly $\mathbf{Y}^{(B)}$, $\mathbf{Z}^{(B)}$ and $\mathcal{E}^{(B)}$ are determined by the vectors $\bar{\mathbf{B}}$ and $\nabla \times \bar{\mathbf{B}}$ and must have the structures $c_1 \bar{\mathbf{B}} + c_2 \nabla \times \bar{\mathbf{B}}$ with two scalar coefficients c_ν , $\nu = 1, 2$. However, assumptions (60) could only be justified if the ratios $c_\nu^{(Y)}/c_\nu^{(\mathcal{E})}$ and $c_\nu^{(Z)}/c_\nu^{(\mathcal{E})}$ of the coefficients $c_\nu^{(Y)}$, $c_\nu^{(Z)}$ and $c_\nu^{(\mathcal{E})}$ for $\mathbf{Y}^{(B)}$, $\mathbf{Z}^{(B)}$ and $\mathcal{E}^{(B)}$, respectively, were independent of ν . We see no reason for that. Some way out could consist in splitting $\mathbf{Y}^{(B)}$, $\mathbf{Z}^{(B)}$ and $\mathcal{E}^{(B)}$ in terms which correspond either to $c_1 \bar{\mathbf{B}}$ or to $c_2 \nabla \times \bar{\mathbf{B}}$ and to formulate ansatzes like (60) for each term separately.

Even if we accept the assumptions (60) the question remains about realistic values of τ_Y and τ_Z , that is, which value of τ_* should be inserted in the results.

Let us point out a conflict of the results of approach (iia) with those of approach (i). Consider the special case in which the nonlinear terms \mathbf{G} and $\tilde{\mathbf{T}}$ tend to zero, that is, $\mathbf{Z} \rightarrow \mathbf{0}$, or $\tau_Z \rightarrow \infty$. We have to require that then results of approach (iia) like (63) or (64) turn into the corresponding results of approach (i). There is indeed some similarity of the results (63) and (64) with the results (29) and (33), or (50) and (55) of approach (i) in the limit $q, p \rightarrow \infty$. However, (63) and (64) do not reflect the fact indicated by (29) and (50), or (32) and (54), that a_{ij} and b_{ijk} , or α and β , in this limit depend on correlations between the components of \mathbf{u} at two different times.

According to the above reasoning the results (63) and (64) should at the same time correspond to the results of approach (i) for the limit $q, p \rightarrow 0$, that is, (30) and (34), or (51). This applies obviously at best in a very crude sense. In particular, the dependence of a_{ij} and b_{ijk} , or α and β , on correlations of components of \mathbf{u} in different points in space is not reproduced.

There is another remarkable difference of the results of approaches (iia) and (i). Consider the case of a purely hydrodynamic background turbulence. In approach (i) the quantities a_{ij} and b_{ijk} as given by (25), (29) or (30) depend only on the correlation properties of \mathbf{u} but, even if \mathbf{u} is influenced by the magnetic field, not explicitly on those of \mathbf{b} ; see also section 3.2.5. In the relation (63) of approach (iia), however, in addition to terms with \mathbf{u} also such with \mathbf{b} occur, which need not be negligible beyond the limit of small $\overline{\mathbf{B}}$. In the relations for α given with (64) and (65) terms with \mathbf{b} occur, too. Whereas (64) applies to isotropic turbulence and thus seems reasonable for small $\overline{\mathbf{B}}$ only, (65) can be considered for larger $\overline{\mathbf{B}}$, where this term needs no longer to be small.

Admitting again a magnetohydrodynamic background turbulence we further note that the times $\tau^{(\alpha u)}$ and $\tau^{(\alpha b)}$ in relation (55) for α , which has been derived in approach (i), need not coincide. In the corresponding result (64) of approach (iia), however, only one comparable time, τ_* , occurs.

Since there is hardly any reason to doubt in the results of approach (i) we conclude that those of approach (iia) are, at least in the range of the validity of approach (i), not completely correct. This, of course, calls also results of approach (iia) for more general cases into question.

We further remark that there is no straightforward extension of approach (iia) as presented above to the case of turbulence in a rotating system. In this case, the equation that occurs instead of (57) contains in addition to \mathcal{E} , which is defined by a product of \mathbf{u} and \mathbf{b} , both taken at the same point, also a term with a product of \mathbf{u} and \mathbf{b} taken at different points. Then \mathcal{E} can no longer be determined on the simple way used above. It is to be expected that a_{ij} and b_{ijk} contain correlations of components of \mathbf{u} at different points in space.

Finally, even an extension in that sense can hardly deliver a correct result for a contribution to \mathcal{E} like $-\delta_1(\boldsymbol{\Omega} \cdot \nabla)\overline{\mathbf{B}}$ in (8), because of $(\boldsymbol{\Omega} \cdot \nabla)\overline{\mathbf{B}} = -\boldsymbol{\Omega} \times (\nabla \times \overline{\mathbf{B}}) + \nabla(\boldsymbol{\Omega} \cdot \overline{\mathbf{B}})$ often discussed as “ $\boldsymbol{\Omega} \times \mathbf{J}$ -effect”. In approach (i) this contribution results, at least in the case of homogeneous turbulence, from a part of the correlation tensor Q_{ij} of \mathbf{u} which is odd in the difference τ of the times at which the components of \mathbf{u} are taken, see appendix C. This part, however, does not occur at all in approach (iia), which considers only correlations of the components of \mathbf{u} at the same time.

The relation (64) for α is often used as a starting point for investigations on α -quenching, e.g., in Blackman and Field (2002) or in Brandenburg and Subramanian (2005a, b). We point out that this relation applies only for isotropic turbulence, that is, in the limit $\overline{\mathbf{B}} \rightarrow \mathbf{0}$. Investigations on α -quenching carried out within the framework of approach (iia) should rather start from a relation like (65). On this level, of course, instead of the simple relations for $\langle \mathbf{b} \cdot (\nabla \times \mathbf{b}) \rangle$ used in the mentioned investigations, which have been concluded from the magnetic helicity balance, corresponding relations for quantities like $\langle \mathbf{b} \times (\mathbf{e}_B \cdot \nabla) \mathbf{b} \rangle \cdot \mathbf{e}_B$ are needed. Moreover, as we know from the above comparison with approach (i), in the case of a purely hydrodynamic background turbulence the existence of a magnetic contribution to α and so its role in α -quenching is questionable.

4.2. A spectral τ -approach (approach (iib))

Let us consider the approach used in the articles by Rädler *et al.* (2003) and by Rogachevskii and Kleeorin (2003, 2004). It has been extensively repeated in the survey article by Brandenburg and Subramanian (2005a), who name it “minimal τ -approximation (MTA)”. As mentioned above we exclude here again any mean motion and restrict ourselves to a non-rotating fluid, that is $\bar{\mathbf{U}} = \bar{\boldsymbol{\Omega}} = \mathbf{0}$.

4.2.1. Outline of approach (iib). In view of the determination of \mathcal{E} the attention is first focused on the cross-correlation tensor χ_{ij} defined by

$$\chi_{ij}(\mathbf{x}, \boldsymbol{\xi}) = \left\langle u_i \left(\mathbf{x} + \frac{\boldsymbol{\xi}}{2} \right) b_j \left(\mathbf{x} - \frac{\boldsymbol{\xi}}{2} \right) \right\rangle. \quad (66)$$

Here and in what follows, all quantities are considered at the same time. For the sake of simplicity we drop the argument t without ignoring the dependence on t .

In addition to χ_{ij} other correlation tensors are of interest, which we first define in the general form

$$\Phi_{ij}(\mathbf{v}, \mathbf{w}; \mathbf{x}, \boldsymbol{\xi}) = \left\langle v_i \left(\mathbf{x} + \frac{\boldsymbol{\xi}}{2} \right) w_j \left(\mathbf{x} - \frac{\boldsymbol{\xi}}{2} \right) \right\rangle. \quad (67)$$

We use a Fourier transformation with respect to the space coordinates only, $F(\mathbf{x}) = \int \hat{F}(\mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{x}) d^3k$. As explained in appendix D the Fourier transform $\hat{\Phi}_{ij}$ of Φ_{ij} with respect to $\boldsymbol{\xi}$ can be represented in the form

$$\hat{\Phi}_{ij}(\mathbf{v}, \mathbf{w}; \mathbf{x}, \mathbf{k}) = \int \left\langle \hat{v}_i \left(\mathbf{k} + \frac{\mathbf{K}}{2} \right) \hat{w}_j \left(-\mathbf{k} + \frac{\mathbf{K}}{2} \right) \right\rangle \exp(i\mathbf{K} \cdot \mathbf{x}) d^3K. \quad (68)$$

For the calculation of \mathcal{E} it is sufficient to know the antisymmetric part of the tensor χ_{ij} , that is, the vector $\boldsymbol{\chi}$ defined by $\chi_i = \epsilon_{ijk} \chi_{jk}$. We introduce its Fourier transform with respect to $\boldsymbol{\xi}$,

$$\hat{\chi}_i(\mathbf{x}, \mathbf{k}) = \epsilon_{ijk} \hat{\chi}_{jk}(\mathbf{x}, \mathbf{k}). \quad (69)$$

Then we have

$$\mathcal{E}(\mathbf{x}) = \int \hat{\chi}(\mathbf{x}, \mathbf{k}) d^3k. \quad (70)$$

Let us consider the time derivative of $\hat{\chi}_{ij}$,

$$\partial_t \hat{\chi}_{ij}(\mathbf{x}, \mathbf{k}) = \int \left\langle \partial_t \hat{u}_i \left(\mathbf{k} + \frac{\mathbf{K}}{2} \right) \hat{b}_j \left(-\mathbf{k} + \frac{\mathbf{K}}{2} \right) + \hat{u}_i \left(\mathbf{k} + \frac{\mathbf{K}}{2} \right) \partial_t \hat{b}_j \left(-\mathbf{k} + \frac{\mathbf{K}}{2} \right) \right\rangle \exp(i\mathbf{K} \cdot \mathbf{x}) d^3K. \quad (71)$$

We consider \mathcal{E} and the quantities necessary for its determination at a given point and choose the coordinate system such that there $\mathbf{x} = \mathbf{0}$. In that sense we assume for the calculation of $\partial_t \hat{u}_i$ and $\partial_t \hat{b}_j$ simply

$$\bar{B}_i(\mathbf{x}, t) = B_i + B_{ij} x_j \quad (72)$$

with constant B_i and B_{ij} . Then we obtain with the equations (4), (14) and (15), reduced to $\mathbf{\Omega} = \bar{\mathbf{U}} = \mathbf{0}$,

$$\begin{aligned} \partial_t \hat{u}_j(\mathbf{k}) &= -\nu k^2 \hat{u}_j(\mathbf{k}) + \frac{1}{\mu_0} \left(i B_l k_l \hat{b}_j(\mathbf{k}) + B_{jl} \hat{b}_l(\mathbf{k}) - 2 B_{lm} \frac{k_j k_l}{k^2} \hat{b}_m(\mathbf{k}) - B_{lm} k_l \frac{\partial \hat{b}_j(\mathbf{k})}{\partial k_m} \right) \\ &\quad + \hat{f}_j(\mathbf{k}) + \hat{T}_j(\mathbf{k}) \\ \partial_t \hat{b}_k(\mathbf{k}) &= -\eta k^2 \hat{b}_k(\mathbf{k}) + i B_l k_l \hat{u}_k(\mathbf{k}) - B_{kl} \hat{u}_l(\mathbf{k}) - B_{lm} k_l \frac{\partial \hat{u}_k(\mathbf{k})}{\partial k_m} + \hat{G}_k(\mathbf{k}). \end{aligned} \quad (73)$$

We insert this into the integral (71) and split it then into a sum of integrals of the type

$$\int F(\mathbf{k}, \mathbf{K}) \left\langle \hat{u}_i \left(\mathbf{k} + \frac{\mathbf{K}}{2} \right) \hat{u}_k \left(\mathbf{k} - \frac{\mathbf{K}}{2} \right) \right\rangle \exp(i \mathbf{K} \cdot \mathbf{x}) d^3 K. \quad (74)$$

We may expand F in a power series with respect to \mathbf{K} and rewrite each factor K_l into a derivative $\nabla_l = \partial/\partial x_l$ of the integral. Since we intend to ignore all contributions to \mathcal{E} with higher than first-order spatial derivatives of mean quantities we do the same with respect to $\hat{\chi}_{ij}$. In this context, of course, B_{il} has to be considered as derivatives of the mean quantity B_i . Using then (69) we obtain

$$\partial_t \hat{\chi} = -(\nu + \eta) k^2 \hat{\chi} + i(\nu - \eta)(\mathbf{k} \cdot \nabla) \hat{\chi} + \hat{\mathbf{I}} + \hat{\mathbf{Y}} + \hat{\mathbf{Z}} \quad (75)$$

with

$$\begin{aligned} \hat{I}_i &= \epsilon_{ijk} \left(-i B_l k_l \hat{u}_{jk}(\mathbf{x}, \mathbf{k}) + \frac{1}{2} B_l \nabla_l \hat{u}_{jk}(\mathbf{x}, \mathbf{k}) - B_{kl} \hat{u}_{jl}(\mathbf{x}, \mathbf{k}) \right. \\ &\quad - \frac{1}{2} B_{lm} k_l \frac{\partial \hat{u}_{jk}(\mathbf{x}, \mathbf{k})}{\partial k_m} - i B_{lm} k_l x_m \hat{u}_{jk}(\mathbf{x}, \mathbf{k}) \\ &\quad + \frac{1}{\mu_0} \left(i B_l k_l \hat{b}_{jk}(\mathbf{x}, \mathbf{k}) + \frac{1}{2} B_l \nabla_l \hat{b}_{jk}(\mathbf{x}, \mathbf{k}) + B_{jl} \hat{b}_{lk}(\mathbf{x}, \mathbf{k}) - 2 B_{lm} \frac{k_j k_l}{k^2} \hat{b}_{mk}(\mathbf{x}, \mathbf{k}) \right. \\ &\quad \left. \left. - \frac{1}{2} B_{lm} k_l \frac{\partial \hat{b}_{jk}(\mathbf{x}, \mathbf{k})}{\partial k_m} + i B_{lm} k_l x_m \hat{b}_{jk}(\mathbf{x}, \mathbf{k}) \right) \right) \\ \hat{Y}_i &= \epsilon_{ijk} \hat{\Phi}_{jk}(\tilde{\mathbf{f}}, \mathbf{b}; \mathbf{x}, \mathbf{k}) \\ \hat{Z}_i &= \epsilon_{ijk} \left(\hat{\Phi}_{jk}(\tilde{\mathbf{T}}, \mathbf{b}; \mathbf{x}, \mathbf{k}) + \hat{\Phi}_{jk}(\mathbf{u}, \mathbf{G}; \mathbf{x}, \mathbf{k}) \right). \end{aligned} \quad (76)$$

The \hat{u}_{jk} and \hat{b}_{jk} are the Fourier transforms of u_{jk} and b_{jk} defined in the sense of (67) by

$$u_{jk}(\mathbf{x}, \xi) = \Phi_{jk}(\mathbf{u}, \mathbf{u}; \mathbf{x}, \xi), \quad b_{jk}(\mathbf{x}, \xi) = \Phi_{jk}(\mathbf{b}, \mathbf{b}; \mathbf{x}, \xi). \quad (77)$$

In contrast to \mathbf{Y} used in approach (iia), $\hat{\mathbf{Y}}$ no longer contains the diffusion terms, that is neither ν nor η , but is only a consequence of the forcing term in the momentum balance.

Like \mathbf{Z} in approach (ia) also $\hat{\mathbf{Z}}$ describes the influence of triple correlations of \mathbf{u} and \mathbf{b} on \mathcal{E} .

We split now \mathcal{E} and likewise $\hat{\chi}$, $\hat{\mathbf{I}}$, $\hat{\mathbf{Y}}$ and $\hat{\mathbf{Z}}$ into parts $\mathcal{E}^{(0)}$, $\hat{\chi}^{(0)}$, $\hat{\mathbf{I}}^{(0)}$, $\hat{\mathbf{Y}}^{(0)}$ and $\hat{\mathbf{Z}}^{(0)}$ independent of $\bar{\mathbf{B}}$ and remaining parts $\mathcal{E}^{(B)}$, $\hat{\chi}^{(B)}$, $\hat{\mathbf{I}}^{(B)}$, $\hat{\mathbf{Y}}^{(B)}$ and $\hat{\mathbf{Z}}^{(B)}$. We have, of course, $\hat{\mathbf{I}}^{(0)} = \mathbf{0}$, that is, $\hat{\mathbf{I}}^{(B)} = \hat{\mathbf{I}}$. Again we do not deal with $\mathcal{E}^{(0)}$. In view of $\mathcal{E}^{(B)}$ we conclude from (75) that

$$\partial_t \hat{\chi}^{(B)} = -(\nu + \eta)k^2 \hat{\chi}^{(B)} + i(\nu - \eta)(\mathbf{k} \cdot \nabla) \hat{\chi}^{(B)} + \hat{\mathbf{I}} + \hat{\mathbf{Y}}^{(B)} + \hat{\mathbf{Z}}^{(B)}. \quad (78)$$

In analogy to (60) we introduce here the assumptions

$$\mathbf{Y}^{(B)}(\mathbf{x}, \mathbf{k}) = -\hat{\chi}^{(B)}(\mathbf{x}, \mathbf{k})/\tau_Y(k), \quad \mathbf{Z}^{(B)}(\mathbf{x}, \mathbf{k}) = -\hat{\chi}^{(B)}(\mathbf{x}, \mathbf{k})/\tau_Z(k) \quad (79)$$

with positive functions $\hat{\tau}_Y$ and $\hat{\tau}_Z$ of k . In Rädler *et al.* (2003) it has not been distinguished between the two terms $\hat{\mathbf{Y}}^{(B)}$ and $\hat{\mathbf{Z}}^{(B)}$, although they are of quite different origin. For their sum an ansatz like that for $\hat{\mathbf{Y}}^{(B)}$ or $\hat{\mathbf{Z}}^{(B)}$ is used. This is, however, of minor interest for the following discussions.

Starting from (78) and using (79) we obtain

$$\partial_t \hat{\chi}^{(B)} + \frac{1}{\hat{\tau}_*} \hat{\chi}^{(B)} = i(\nu - \eta)(\mathbf{k} \cdot \nabla) \hat{\chi}^{(B)} + \hat{\mathbf{I}}, \quad \frac{1}{\hat{\tau}_*} = (\nu + \eta)k^2 + \frac{1}{\hat{\tau}_Y} + \frac{1}{\hat{\tau}_Z}. \quad (80)$$

We recall that all contributions to \mathcal{E} with higher than first-order derivatives of mean quantities should be ignored. This applies, of course, to $\hat{\chi}^{(B)}$ and $\hat{\mathbf{I}}$, too. In that sense we conclude from (80) that

$$\partial_t(\mathbf{k} \cdot \nabla) \hat{\chi}^{(B)} + \frac{1}{\hat{\tau}_*}(\mathbf{k} \cdot \nabla) \hat{\chi}^{(B)} = (\mathbf{k} \cdot \nabla) \hat{\mathbf{I}}. \quad (81)$$

In view of \mathcal{E} at $\mathbf{x} = \mathbf{0}$ we consider also the equations (80) and (81) at $\mathbf{x} = \mathbf{0}$. They are then differential equations for $\hat{\chi}^{(B)}$ and $(\mathbf{k} \cdot \nabla) \hat{\chi}^{(B)}$ with respect to t . Solving them in the same way as we did it with (61), assuming here, too, $t_0 \rightarrow -\infty$ and sufficiently small variations of $\hat{\mathbf{I}}$ in time, we find

$$\hat{\chi}^{(B)} = (\hat{\mathbf{I}} + i(\nu - \eta)(\mathbf{k} \cdot \nabla) \hat{\mathbf{I}} \hat{\tau}_*) \hat{\tau}_*, \quad (82)$$

and, together with (76),

$$\begin{aligned} \hat{\chi}_i^{(B)} = & \epsilon_{ijk} \left(\left(\frac{1}{2} \nabla_p \hat{u}_{jk} - ik_p \hat{u}_{jk} + \frac{1}{\mu_Q} \left(\frac{1}{2} \nabla_p \hat{b}_{jk} + ik_p \hat{b}_{jk} \right) \right. \right. \\ & + (\nu - \eta) k_l k_p \nabla_l \left(\hat{u}_{jk} - \frac{1}{\mu_Q} \hat{b}_{jk} \right) \hat{\tau}_*(k) \left. \right) \hat{\tau}_*(k) B_p \\ & - \left(\hat{u}_{jq} \delta_{kp} + \frac{1}{2} k_p \frac{\partial}{\partial k_q} \hat{u}_{jk} - \frac{1}{\mu_Q} \left(\hat{b}_{qk} \delta_{jp} - \frac{1}{2} k_p \frac{\partial}{\partial k_q} \hat{b}_{jk} - 2 \frac{k_j k_p}{k^2} \hat{b}_{qk} \right) \right. \\ & \left. - (\nu - \eta) k_p k_q \left(\hat{u}_{jk} - \frac{1}{\mu_Q} \hat{b}_{jk} \right) \hat{\tau}_*(k) \right) \hat{\tau}_*(k) B_{pq}, \end{aligned} \quad (83)$$

again with all quantities taken at $\mathbf{x} = \mathbf{0}$.

With this result we can calculate \mathcal{E} according to (70). Admitting again arbitrary \mathbf{x} and writing the result in the form (6), that is, $\mathcal{E}_i = \mathcal{E}_i^{(0)} + a_{ij}\bar{B}_j + b_{ijk}\partial\bar{B}_j/\partial x_k$, we have

$$\begin{aligned}
 a_{ip} &= \epsilon_{ijk} \int \left(\frac{1}{2} \nabla_p \hat{u}_{jk}(\mathbf{x}, \mathbf{k}) - ik_p \hat{u}_{jk}(\mathbf{x}, \mathbf{k}) + \frac{1}{\mu_Q} \left(\frac{1}{2} \nabla_p \hat{b}_{jk}(\mathbf{x}, \mathbf{k}) + ik_p \hat{b}_{jk}(\mathbf{x}, \mathbf{k}) \right) \right. \\
 &\quad \left. + (\nu - \eta) k_l k_p \nabla_l \left(\hat{u}_{jk}(\mathbf{x}, \mathbf{k}) - \frac{1}{\mu_Q} \hat{b}_{jk}(\mathbf{x}, \mathbf{k}) \right) \hat{\tau}_*(k) \right) \hat{\tau}_*(k) d^3 k \\
 b_{ipq} &= -\epsilon_{ijk} \int \left(\hat{u}_{jq}(\mathbf{x}, \mathbf{k}) \delta_{kp} + \frac{1}{2} k_p \frac{\partial}{\partial k_q} \hat{u}_{jk}(\mathbf{x}, \mathbf{k}) \right. \\
 &\quad - \frac{1}{\mu_Q} \left(\hat{b}_{qk}(\mathbf{x}, \mathbf{k}) \delta_{jp} - \frac{1}{2} k_p \frac{\partial}{\partial k_q} \hat{b}_{jk}(\mathbf{x}, \mathbf{k}) - 2 \frac{k_j k_p}{k^2} \hat{b}_{qk}(\mathbf{x}, \mathbf{k}) \right) \\
 &\quad \left. - (\nu - \eta) k_p k_q \left(\hat{u}_{jk}(\mathbf{x}, \mathbf{k}) - \frac{1}{\mu_Q} \hat{b}_{jk}(\mathbf{x}, \mathbf{k}) \right) \hat{\tau}_*(k) \right) \hat{\tau}_*(k) d^3 k.
 \end{aligned} \tag{84}$$

These relations are by the convolution theorem equivalent to

$$\begin{aligned}
 a_{ip} &= \frac{\epsilon_{ijk}}{(2\pi)^3} \left(\int_{\infty} \left(\left\langle u_j(\mathbf{x} + \xi/2, t) \frac{\partial u_k(\mathbf{x} - \xi/2, t)}{\partial x_p} \right\rangle \right. \right. \\
 &\quad \left. \left. + \frac{1}{\mu_Q} \left\langle \frac{\partial b_j(\mathbf{x} + \xi/2, t)}{\partial x_p} b_k(\mathbf{x} - \xi/2, t) \right\rangle \right) \tau_*(\xi) d^3 \xi \right. \\
 &\quad \left. - (\nu - \eta) \frac{\partial}{\partial x_l} \int_{\infty} \frac{\partial^2}{\partial \xi_l \partial \xi_p} \left(\langle u_j(\mathbf{x} + \xi/2, t) u_k(\mathbf{x} - \xi/2, t) \rangle \right. \right. \\
 &\quad \left. \left. - \frac{1}{\mu_Q} \langle b_j(\mathbf{x} + \xi/2, t) b_k(\mathbf{x} - \xi/2, t) \rangle \right) \check{\tau}_*^2(\xi) d^3 \xi \right) \\
 b_{ipq} &= -\frac{\epsilon_{ijk}}{(2\pi)^3} \left(\int_{\infty} \left(\langle u_j(\mathbf{x} + \xi/2, t) u_q(\mathbf{x} - \xi/2, t) \rangle \delta_{kp} \right. \right. \\
 &\quad \left. \left. - \frac{\xi_q}{2} \frac{\partial}{\partial \xi_p} \langle u_j(\mathbf{x} + \xi/2, t) u_k(\mathbf{x} - \xi/2, t) \rangle \right. \right. \\
 &\quad \left. \left. - \frac{1}{\mu_Q} \left(\langle b_q(\mathbf{x} + \xi/2, t) b_k(\mathbf{x} - \xi/2, t) \rangle \delta_{jp} \right. \right. \right. \\
 &\quad \left. \left. + \frac{\xi_q}{2} \frac{\partial}{\partial \xi_p} \langle b_j(\mathbf{x} + \xi/2, t) b_k(\mathbf{x} - \xi/2, t) \rangle \right. \right. \\
 &\quad \left. \left. - 2 \frac{\partial^2}{\partial \xi_j \partial \xi_p} (\Delta_{\xi}^{-1} \langle b_q(\mathbf{x} + \xi/2, t) b_k(\mathbf{x} - \xi/2, t) \rangle) \right) \right) \tau_*(\xi) d^3 \xi \\
 &\quad + (\nu - \eta) \int_{\infty} \frac{\partial^2}{\partial \xi_q \partial \xi_p} \left(\langle u_j(\mathbf{x} + \xi/2, t) u_k(\mathbf{x} - \xi/2, t) \rangle \right. \\
 &\quad \left. - \frac{1}{\mu_Q} \langle b_j(\mathbf{x} + \xi/2, t) b_k(\mathbf{x} - \xi/2, t) \rangle \right) \check{\tau}_*^2(\xi) d^3 \xi.
 \end{aligned} \tag{85}$$

For clarity we have inserted again the arguments t . The quantities τ_* and $\check{\tau}_*$ are given by

$$\begin{aligned}\tau_*(\xi) &= \int \hat{\tau}_*(k) \exp(\mathbf{i}\mathbf{k} \cdot \boldsymbol{\xi}) d^3k = \frac{4\pi}{\xi} \int_0^\infty \hat{\tau}_*(k) \sin(k\xi) k dk \\ \check{\tau}_*^2(\xi) &= \int \hat{\tau}_*^2(k) \exp(\mathbf{i}\mathbf{k} \cdot \boldsymbol{\xi}) d^3k = \frac{1}{(2\pi)^3} \int_\infty^\infty \tau_*(|\boldsymbol{\xi} - \boldsymbol{\xi}'|) \tau_*(\xi') d^3\xi'.\end{aligned}\quad (86)$$

Note that τ_* has the dimension $\text{time} \times \text{length}^{-3}$, and $\check{\tau}_*$ the dimension $\text{time} \times \text{length}^{-3/2}$. As for Δ_ξ^{-1} we refer to the definition (47).

Let us again consider the special case of homogeneous isotropic turbulence. Then we have again $\boldsymbol{\mathcal{E}}^{(0)} = \mathbf{0}$ and $a_{ip} = \alpha \delta_{ip}$ and $b_{ipq} = \beta \epsilon_{ipq}$ so that $\boldsymbol{\mathcal{E}}$ takes the form (7), that is $\boldsymbol{\mathcal{E}} = \alpha \bar{\mathbf{B}} - \beta \nabla \times \bar{\mathbf{B}}$. Using (85) we find

$$\begin{aligned}\alpha &= -\frac{1}{3(2\pi)^3} \int_\infty^\infty \left(\left\langle \mathbf{u}\left(\mathbf{x} + \frac{\boldsymbol{\xi}}{2}, t\right) \cdot \left(\nabla \times \mathbf{u}\left(\mathbf{x} - \frac{\boldsymbol{\xi}}{2}, t\right)\right) \right\rangle \right. \\ &\quad \left. - \frac{1}{\mu\varrho} \left\langle \mathbf{b}\left(\mathbf{x} + \frac{\boldsymbol{\xi}}{2}, t\right) \cdot \left(\nabla \times \mathbf{b}\left(\mathbf{x} - \frac{\boldsymbol{\xi}}{2}, t\right)\right) \right\rangle \right) \tau_*(\xi) d^3\xi \\ \beta &= \frac{1}{3(2\pi)^3} \int_\infty^\infty \left\langle \mathbf{u}\left(\mathbf{x} + \frac{\boldsymbol{\xi}}{2}, t\right) \cdot \mathbf{u}\left(\mathbf{x} - \frac{\boldsymbol{\xi}}{2}, t\right) \right\rangle \tau_*(\xi) d^3\xi.\end{aligned}\quad (87)$$

Here we have replaced $\langle \mathbf{b}(\mathbf{x} - \boldsymbol{\xi}/2, t) \cdot (\nabla \times \mathbf{b}(\mathbf{x} + \boldsymbol{\xi}/2, t)) \rangle$, what originally occurs in the derivation from (85), by $\langle \mathbf{b}(\mathbf{x} + \boldsymbol{\xi}/2, t) \cdot (\nabla \times \mathbf{b}(\mathbf{x} - \boldsymbol{\xi}/2, t)) \rangle$. Due to the assumed homogeneity of the turbulence the averages $\langle \dots \rangle$ in (87) do not depend on \mathbf{x} , and we may further replace, e.g., $\langle \mathbf{u}(\mathbf{x} + \boldsymbol{\xi}/2, t) \cdot (\nabla \times \mathbf{u}(\mathbf{x} - \boldsymbol{\xi}/2, t)) \rangle$ with $\langle \mathbf{u}(\mathbf{x}, t) \cdot (\nabla \times \mathbf{u}(\mathbf{x} + \boldsymbol{\xi}, t)) \rangle$. Again, by the reason explained in the context of (31) the integrals $\int_\infty^\infty \dots d^3\xi$ may be replaced with $4\pi \int_0^\infty \dots \xi^2 d\xi$. The $(\nu - \eta)$ terms in (85) are without influence on α and β .

In Rädler *et al.* (2003) the possibility of a non-zero $\boldsymbol{\mathcal{E}}^{(0)}$ has been ignored from the very beginning. In addition, only the limit of small $\bar{\mathbf{B}}$ is considered. So for a comparison, \hat{u}_{ij} and \hat{b}_{ij} in (83) and (84) have to be interpreted as $\hat{u}_{ij}^{(0)}$ and $\hat{b}_{ij}^{(0)}$, the limits of these quantities for $\bar{\mathbf{B}} \rightarrow \mathbf{0}$, and likewise \mathbf{u} and \mathbf{b} in (85) and (87) as $\mathbf{u}^{(0)}$ and $\mathbf{b}^{(0)}$, understood in the same sense.

4.2.2. Comments. Similar to the situation with approach (iia) and the assumptions (60) on $\mathbf{Y}^{(B)}$ and $\mathbf{Z}^{(B)}$ the results of the above derivation depend crucially on the assumptions (79) on $\hat{\mathbf{Y}}^{(B)}$ and $\hat{\mathbf{Z}}^{(B)}$ describing the influences of the forcing and of the triple correlations of \mathbf{u} and \mathbf{b} . So here we have essentially to repeat the comment made in 4.1.2 in view of the assumptions (60) used in approach (iia). Here, too, we see no rigorous way to justify assumption (79). Moreover, the choice of $\hat{\tau}_Y$ and $\hat{\tau}_Z$ remains open.

A very strange aspect of the above procedure is that it needs as input no other correlation tensors than those with components of \mathbf{u} or \mathbf{b} taken at the same time.

Let us again consider the case in which the nonlinear terms \mathbf{G} and \mathbf{T} tend to zero. This implies $\hat{\mathbf{Z}}^{(B)} \rightarrow \mathbf{0}$, or $\tau_Z \rightarrow \infty$. In this case, the results of approach (iib) have to coincide with those of approach (i).

As can be seen, e.g., from (80) the results of approach (iib) depend in general on both ν and η . This is clearly in conflict with the fact that those of approach (i) for purely hydrodynamic turbulence do not depend on ν .

In the limit $q \rightarrow \infty$, approach (i) delivered us the quantities a_{ij} and b_{ijk} in the form (29). These are far from coinciding with the results (85) obtained in approach (iib). A striking discrepancy is that in approach (i), apart from the independence of the results on ν , only correlations of the components of \mathbf{u} at the same point in space but at different times occur, in approach (iib) only correlations at different points in space but at the same time. At first glance this discrepancy seems to disappear in the case of homogeneous isotropic statistically steady turbulence, in which the results of approach (i) for α and β given by (32) can be rewritten in the forms (B9) and (B12), which correspond to (87). While however the τ_c in (B9) and (B12) may differ from each other the τ_* in (32) have to coincide.

In the limit $q \rightarrow 0$, approach (i) delivers us the relations (30) for a_{ij} and b_{ijk} . They are in so far similar to (85) as in both cases only correlations of the components of \mathbf{u} at the same time occur. Nevertheless a coincidence of the two kinds of results would require very special properties of the factor τ_* in (85). We see no reasons for such properties. When putting on trial $\hat{\mathbf{Y}}^{(B)} = \mathbf{0}$ (Brandenburg and Subramanian (2005a) consider $\hat{\mathbf{Y}}^{(B)}$ as negligible) we arrive at expressions for α and β which differ from (34) only in so far as $\eta + \nu$ occurs instead of η . A view at the first of the equations (73) might suggest to put $\hat{\mathbf{Y}}^{(B)} = \nu k^2 \hat{\chi}^{(B)}$. Then we obtain indeed α and β in full agreement with (34). Nevertheless $\hat{\mathbf{Y}}^{(B)} = \nu k^2 \hat{\chi}^{(B)}$ removes ν only from one of two places at which it occurs in (80), and even if it does no longer occur in α and β it will do so in other mean-field coefficients. Hence, we believe that an essential reason for shortcomings of approach (iib) is a treatment of the term $\hat{\mathbf{Y}}^{(B)}$, which is in general not justified.

In the discussion of the results of approach (iia), we have pointed out a difference to those of approach (i), which becomes visible in the special case of purely hydrodynamic background turbulence. This difference occurs here, too. In this case in approach (i) the quantities a_{ij} and b_{ijk} , or α and β , can always be represented by the correlation properties of \mathbf{u} alone. In approach (iib) however, beyond the limit of small $\bar{\mathbf{B}}$, the expressions for these quantities except β contain both \mathbf{u} and \mathbf{b} .

We see that approach (iib) to \mathcal{E} delivers results which even under comparable conditions deviate from the results of approach (i), which we consider as correct. Therefore, also beyond the range of validity of approach (i) and beyond the case of non-rotating turbulence, all results derived in the framework of approach (iib) have to be considered with caution. Their details are in general not correct.

If we admit a Coriolis force, that is $\boldsymbol{\Omega} \neq \mathbf{0}$, or a mean motion, $\bar{\mathbf{U}} \neq \mathbf{0}$, there is a coupling between the symmetric and antisymmetric parts of $\hat{\chi}_{ij}$, and the calculation of $\hat{\chi}$ is more complex. By the reason already discussed in section 4.1.2 it is then hard to imagine that an extension of approach (iib) to the case $\boldsymbol{\Omega} \neq \mathbf{0}$ can describe, e.g., the “ $\boldsymbol{\Omega} \times \mathbf{J}$ -effect” correctly. Here a view on the “ $\boldsymbol{\Omega} \times \mathbf{J}$ -dynamo” is of some interest. Its existence depends on a non-zero value of a specific coefficient. In Rädler *et al.* (2003), equations (130) and (132), it is given by $\delta_0 - \kappa_0$, and for hydrodynamic

background turbulence it is non-zero as long as $\hat{\tau}_*$ depends on k . According to the calculations by Rädler and Stepanov (2006) done again for hydrodynamic background turbulence and in the second-order correlation approximation, however, this value vanishes in both the limits $q \rightarrow 0$ and $q \rightarrow \infty$ independently of any assumption on the correlation time; see section VD and appendix C of that article.

Rogachevskii and Kleorin (2003), considering a mean motion and proceeding in the spirit of approach (iib), found a contribution to \mathcal{E} proportional to $\mathbf{W} \times (\nabla \times \bar{\mathbf{B}})$, where \mathbf{W} stands for $\nabla \times \bar{\mathbf{U}}$. A contribution which can be interpreted in that sense occurs already in an article by Urpin (1999), though as a result of a not easily comprehensible calculation. The existence of this “ $\mathbf{W} \times \mathbf{J}$ -effect” has been confirmed in the framework of the second-order correlation approximation by Rädler and Stepanov (2006). The simple model of an “ $\mathbf{W} \times \mathbf{J}$ -dynamo” proposed by Rogachevskii and Kleorin (2003) (see also Rogachevskii and Kleorin 2004) works with the coefficients determining this and the accompanying effects as calculated in the spectral τ -approach. It turned out, however, that it fails to work with the coefficients resulting from the second-order correlation approximation for purely hydrodynamic background turbulence, and this applies independently of q (Rädler and Stepanov 2006, Rüdiger and Kitchatinov 2006).

4.3. The τ -approaches in the light of numerical simulations

Quite naturally the question arises whether the correctness or the limits of validity of the τ -approaches can be checked by means of direct numerical simulations. Indeed, Brandenburg and Subramanian (2005b) report on numerical experiments with forced helical magnetohydrodynamic turbulence in a rectangular box aimed at studying α -effect and α -quenching in the framework of approach (iia). In the underlying theory they use the ansatz (60) for $\mathbf{Z}^{(B)}$ (in their notation $\bar{\mathbf{T}}$), standing for triple correlations, but ignore $\mathbf{Y}^{(B)}$, which comprises the effects of viscous and ohmic dissipation and the correlation of the magnetic fluctuations with the external force.

Brandenburg and Subramanian examined the relation between $\mathbf{Z}^{(B)}$ and a dynamo-generated mean field $\bar{\mathbf{B}}$ on the basis of a simulation with a forcing yielding an isotropic background turbulence. Under the assumption of the proportionality of \mathcal{E} and $\bar{\mathbf{B}}$, justified by a weak spatial variation of $\bar{\mathbf{B}}$, the result turned out to be consistent with the ansatz (60) for $\mathbf{Z}^{(B)}$. The simulation confirmed not only the proportionality of $\mathbf{Z}^{(B)}$ and \mathcal{E} , which was, due to the isotropy of the turbulence, to be expected already by symmetry reasons, but delivered also a non-negative τ_Z . The ansatz (60) for $\mathbf{Y}^{(B)}$, which was not considered in this work, is plausible by symmetry reasons, too. So under the considered circumstances the neglect of a non-zero $\mathbf{Y}^{(B)}$ cannot have any other consequence in the result for α than a deviation of the time τ_* from τ_Z .

In a further step, Brandenburg and Subramanian extracted the time τ_* (in their notation τ) from the simulations by making use of the relation (64) for α . Of course, using (64) instead of (65) has generally to be questioned as only the latter is the relevant one beyond the limit $\bar{\mathbf{B}} \rightarrow \mathbf{0}$. The use of (64) can only be justified under the additional assumption that $-(1/3)\langle \mathbf{u} \cdot (\nabla \times \mathbf{u}) \rangle$ and $-(1/3)\langle \mathbf{b} \cdot (\nabla \times \mathbf{b}) \rangle$ are proper proxies of $\langle \mathbf{u} \times (\mathbf{e}_B \cdot \nabla) \mathbf{u} \rangle \cdot \mathbf{e}_B$ and $\langle \mathbf{b} \times (\mathbf{e}_B \cdot \nabla) \mathbf{b} \rangle \cdot \mathbf{e}_B$. In some cases, from the simulations negative values of τ_* emerged (surprisingly enough just for small $\bar{\mathbf{B}}$), which is in contradiction with the idea of a mean electromotive force decaying in the absence of a mean magnetic field. As a way out an ad hoc modification of (64) was introduced

in the form of additional quenching functions, g_K and g_M , in front of both the kinetic and the magnetic contributions to α . This measure cannot be brought into accordance with the derivation of (64), which allows at best the same quenching function for both contributions, that is, a dependence of τ_* on $|\mathbf{B}|$. Interestingly enough, in the relations (52) and (55) for α , derived in approach (i) for magnetohydrodynamic background turbulence in the limit $q, p \rightarrow \infty$, the two times $\tau^{(\alpha u)}$ and $\tau^{(\alpha b)}$ occur, which could be well different from each other. As far as the assumption of magnetohydrodynamic background turbulence, that is, of a small-scale dynamo, is justified, the numerical results support our doubt in the correctness of approach (iia) inferred from the conflict of the latter with approach (i). It is however not clear to us whether all runs correspond to this assumption. In the case of a purely hydrodynamic background turbulence, there is still another doubt in relation (64) for α , which also results from the comparison with approach (i) and concerns the existence of the magnetic contribution to α (see section 4.1.2).

Let us for a moment deviate from approach (iia) and accept the concept employing the two quenching functions g_K and g_M . In order to get enough constraints for their unambiguous determination, results of simulations with different (kinetic and magnetic) forcings were employed. This seems to be again questionable, as only the structures of relations like (64) or (65) can be considered as universal in their validity ranges, but not the specific values of τ_* or of g_K and g_M . These values should in general differ for different realizations of turbulence. As a consequence, they can at best be determined using data from one and the same simulation or from simulations which are equivalent on the level of mean quantities. Examining the visualizations of the employed realizations of turbulence given in Brandenburg and Subramanian (2005b) (figures 10 and 11) it is by no means clear whether this requirement is fulfilled. But even if so, an unambiguous determination of the two quenching factors were then again impossible because the relevant equations (17) of that article would just become linearly dependent.

In view of some support of approach (iia) from numerical studies, we mention also the investigations of Blackman and Field (2002) and Field and Blackman (2002) on the saturation of a mean-field dynamo. Specific results on helicities obtained in this approach, even if using a relation for α like (64) instead of one like (65), are surprisingly well matched to those of numerical simulations by Brandenburg (2001), which are independent of such an approach.

Summing up, we state that a convincing numerical support of the τ -approaches still requires further efforts beyond the work published so far.

In this context, studies of the behavior of passive scalars in a turbulent fluid carried out by Blackman and Field (2003) and by Brandenburg *et al.* (2004) using the τ -approach, too, deserve some interest. Numerical simulations confirm the results derived in the framework of this approximation rather well. This is remarkable since the concerns discussed with respect to the treatment of the kinetic forcing term (see section 4.2.2) apply here, too.

5. An alternative

As explained above, approach (i) is based on approximative solutions of the induction equation (4) for \mathbf{b} . The approximation is defined by the restriction to second-order

correlations in the velocity fluctuations \mathbf{u} . In contrast to this, in the approaches (ii) neither the induction equation nor the momentum equation are really solved. Instead simple assumptions are used for some crucial quantities. These assumptions, however, have proved to be very problematic. We believe that a real progress in the calculation of the mean electromotive force \mathcal{E} beyond the second-order correlation approximation can only be gained on the basis of improved solutions of the induction equation (4). In the expressions for \mathcal{E} obtained in this way the quantities depending on \mathbf{u} can then be specified according to the momentum equation.

5.1. Higher-order correlation approximations

As is well known, it is possible to proceed from the second-order correlation approximation for the mean electromotive force \mathcal{E} to higher-order approximations (Krause and Rädler 1971, 1980). Basically, arbitrarily high orders can be reached. It has been shown that the proposed procedure converges (Krause 1986, Krause and Rädler 1980). However, the calculation of \mathcal{E} in a higher-order approximation requires the knowledge of higher-order correlation tensors and is moreover very tedious. That is why so far only in a few simple cases higher-order results have been given (e.g., Nicklaus and Stix 1988, Carvalho 1992, 1994, Rädler *et al.* 1997).

In appendix E, a procedure is described for the calculation of a_{ij} and b_{ijk} in higher-order correlation approximations. For the sake of simplicity, it is restricted to cases with $\bar{\mathbf{U}} = \mathbf{0}$ and $\nabla \cdot \mathbf{u} = 0$. It turns out that then a_{ij} and b_{ijk} can be represented in the simple form

$$\begin{aligned} a_{ij}(\mathbf{x}, t) &= \epsilon_{ilm} \int_0^\infty \int_\infty^\infty \frac{\partial G(\xi, \tau)}{\partial \xi_j} \overline{u_l(\mathbf{x}, t) v_m(\mathbf{x}, t; -\xi, -\tau)} d^3\xi d\tau \\ b_{ijk}(\mathbf{x}, t) &= \epsilon_{ilm} \int_0^\infty \int_\infty^\infty G(\xi, \tau) \overline{u_l(\mathbf{x}, t) w_{mjk}(\mathbf{x}, t; -\xi, -\tau)} d^3\xi d\tau. \end{aligned} \quad (88)$$

The quantities v_m and w_{mjk} are given as sums of constituents $v_m^{(v)}$ and $w_{mjk}^{(v)}$, where v is the order in \mathbf{u} . Recursions for the calculation of $v_m^{(v)}$ from $v_m^{(v-1)}$ and of $w_{mjk}^{(v)}$ from $w_{mjk}^{(v-1)}$ are available, see appendix E.

Let us add a remark concerning $\mathcal{E}^{(0)}$. As explained above, equation (4) for \mathbf{b} in its original form, that is, without the neglect of \mathbf{G} , with a properly specified \mathbf{u} may have non-decaying solutions even if $\bar{\mathbf{B}}$ is equal to zero. Then $\mathcal{E}^{(0)}$ may be unequal to zero. As already noted in section 3.1.1, in the second-order correlation approximation, when ignoring the influence of the initial \mathbf{b} , no such solutions \mathbf{b} exist and therefore $\mathcal{E}^{(0)}$ vanishes. Hence, if we start from the second-order approximation and then proceed to higher approximations we never find $\mathcal{E}^{(0)}$, though it may well be non-zero. In that sense the iteration procedure is not complete.

5.2. A closure proposal

Instead of approaches of type (ii) to the mean electromotive force \mathcal{E} we propose another one. For the sake of simplicity, we restrict ourselves again to the case $\bar{\mathbf{U}} = \mathbf{0}$ and ignore

the influence of any initial value of \mathbf{b} , that is, consider the limit $t_0 \rightarrow -\infty$. Then equation (4) is equivalent to

$$b_k(\mathbf{x}, t) = (\delta_{kl}\delta_{im} - \delta_{km}\delta_{il}) \int_0^\infty \int_\infty \frac{\partial G(\xi, \tau)}{\partial \xi_l} \left(u_m(\mathbf{x} - \xi, t - \tau) \bar{B}_n(\mathbf{x} - \xi, t - \tau) + (u_m(\mathbf{x} - \xi, t - \tau) b_n(\mathbf{x} - \xi, t - \tau))' \right) d^3\xi d\tau. \quad (89)$$

This leads immediately to

$$\mathcal{E}_i(\mathbf{x}, t) = (\delta_{il}\epsilon_{kmn} - \delta_{kl}\epsilon_{imn}) \int_0^\infty \int_\infty \frac{\partial G(\xi, \tau)}{\partial \xi_l} \left((u_k(\mathbf{x}, t) u_m(\mathbf{x} - \xi, t - \tau)) \bar{B}_n(\mathbf{x} - \xi, t - \tau) - \langle u_k(\mathbf{x}, t) u_m(\mathbf{x} - \xi, t - \tau) b_n(\mathbf{x} - \xi, t - \tau) \rangle \right) d^3\xi d\tau. \quad (90)$$

We write for short

$$\mathcal{E} = \mathcal{E}^{(SOCA)} + \mathcal{E}^{(uub)}, \quad (91)$$

where $\mathcal{E}^{(SOCA)}$ is the result for \mathcal{E} obtained in the second-order correlation approximation, that is, with (90) if the part of the integrand with b_n is ignored, and $\mathcal{E}^{(uub)}$ the contribution to \mathcal{E} defined by this part.

Our proposal for a new approach to \mathcal{E} consists in its simplest form in putting

$$\mathcal{E}^{(uub)} = -f \mathcal{E}^{(SOCA)} \quad (92)$$

with some factor f , which will be discussed below. Then we have

$$\mathcal{E} = (1 - f) \mathcal{E}^{(SOCA)}. \quad (93)$$

The ansatz (92) corresponds in a formal sense to the ansatzes (60) or (79) used in the approaches (ii). As the time τ_* in these cases, the factor f remains at first undetermined. In order to find rough statements or estimates for the factor f , a look on the results of higher-order approximations reported above is useful. These results suggest that, e.g., in the limit $q \rightarrow 0$ the factor f has the form $f = c_1 Rm$ with some positive c_1 for small Rm , and $f = c_1 Rm + c_2 Rm^2 + \dots$ for higher Rm . In the limit $q \rightarrow \infty$, the factor f should have the form $f = c_1 St$ for small St , and the form $f = c_1 St + c_2 St^2 + \dots$ for higher St . Since $f \rightarrow 0$ as $Rm \rightarrow 0$ or $St \rightarrow 0$, respectively, the ansatz (92) trivially satisfies the requirement that higher approaches should include the second-order correlation approximation in overlapping parts of their respective validity regions.

In this context the α -effect calculations for the Karlsruhe experiment are of particular interest (Rädler *et al.* 1997, 1998, 2002). They have been done for the case $\bar{\mathbf{U}} = \mathbf{0}$ and steady flows \mathbf{u} , that is $q = 0$, with arbitrary Rm . The flows have the form $\mathbf{u} = \mathbf{u}_\perp + \mathbf{u}_\parallel$, where \mathbf{u}_\perp and \mathbf{u}_\parallel are, with respect to a proper Cartesian coordinate system (x, y, z) , defined by $\mathbf{u}_\perp = (u_x(x, y), u_y(x, y), 0)$ and $\mathbf{u}_\parallel = (0, 0, u_z(x, y))$, and are periodic in x and y . Two different magnetic Reynolds numbers Rm_\perp and Rm_\parallel are introduced characterizing the magnitudes of \mathbf{u}_\perp and \mathbf{u}_\parallel , respectively. If the flow described by \mathbf{u} shows helical structures as, e.g., in the case of the so-called Roberts flow, an α -effect occurs such that $\mathcal{E} = \alpha_\perp \bar{\mathbf{B}}_\perp$, with some coefficient α_\perp , and $\bar{\mathbf{B}}_\perp$ defined analogously to \mathbf{u}_\perp ; contributions to \mathcal{E} with derivatives of $\bar{\mathbf{B}}$ are here ignored. In the second-order correlation approximation the result for α_\perp , in this case called $\alpha_\perp^{(SOCA)}$, reads

$\alpha_{\perp}^{(SOCA)} = \alpha_0 Rm_{\perp} Rm_{\parallel}$, with some α_0 depending on the geometrical properties, but not on the magnitude of the flow. In the general case, it turns out that $\alpha_{\perp} = \alpha_0 Rm_{\perp} Rm_{\parallel} \phi(Rm_{\perp})$ with the same α_0 as before. The function ϕ satisfies $\phi(0) = 1$, decays monotonically with growing Rm_{\perp} and tends to zero as $Rm_{\perp} \rightarrow \infty$; further properties and numerical values of ϕ are given in some of the articles mentioned (Rädler *et al.* 1997, 2002). When writing in the sense of (93) $\alpha_{\perp} = (1-f)\alpha_{\perp}^{(SOCA)}$ we have $f = 1 - \phi$ and therefore $f \rightarrow 0$ as $Rm_{\perp} \rightarrow 0$, and $f \rightarrow 1$ if $Rm_{\perp} \rightarrow \infty$.

6. Summary

The article provides a critical view on different analytical ways to determine the mean electromotive force \mathcal{E} in mean-field electrodynamics. First, some of the findings gained with the traditional approach, reduced to the second-order correlation approximation or approach (i), are summarized (section 3). In this context also some new results concerning the case of magnetohydrodynamic background turbulence are given (section 3.2). Then the essentials of two versions of the τ -approach, or approach (ii), are represented, that is, of the simple τ -approach, or approach (iia), as used by Vainshtein and Kichatinov (1983), Blackman and Field (2002) and Brandenburg and Subramanian (2005b), and of the spectral τ -approach, or approach (iib), used by Rädler *et al.* (2003) and by Rogachevskii and Kleeorin (2003, 2004) (section 4).

Approach (i) in its original form, applying to purely hydrodynamic background turbulence, is based on solutions of the induction equation for the fluctuations \mathbf{b} simplified by some approximation. In the case of magnetohydrodynamic background turbulence, in addition solutions of the momentum balance for \mathbf{u} , again in some approximation, are used. In the approaches (ii) these equations are not really solved. Instead, some assumptions on crucial quantities are introduced. There is hardly any doubt in the correctness of the results of approach (i) in the range of its applicability, which is well defined at least by sufficient conditions. It is not surprising that the approaches (ii) deliver the same vectorial structures of the contributions to \mathcal{E} as approach (i), for these are already determined by elementary symmetry principles. The results of the two types of approaches differ however in the dependence of the coefficients of the individual contributions to \mathcal{E} on the correlation properties of the velocity fluctuations \mathbf{u} . The discrepancies of the two approaches as well as strange aspects or shortcomings of the approaches (ii) are discussed in detail (sections 4.1.2 and 4.2.2). Unless any overlap of the ranges of applicability of the approaches (i) and (ii) can be excluded, what would be very surprising, we have to conclude that the approaches (ii) are in some conflict with the basic equations and not all of their results can be taken for granted. For instance, the magnetic contribution to the α -effect occurs in the approaches (ii) not only in the case of a magnetohydrodynamic background turbulence but for purely hydrodynamic background turbulence, too, where this contribution should not exist. In addition, in the first case it is the same ‘‘correlation time’’ which occurs with the kinetic and the magnetic contributions, what seems to be in conflict with numerical simulations (section 4.3). Further the dependence of α and β on the magnetic diffusivity η and the kinematic viscosity ν which occurs in approach (iib) is not correct. Moreover, it is doubtful whether the approaches (ii) can describe, e.g., the $\mathbf{\Omega} \times \mathbf{J}$ -effect correctly.

There is no hint that the mentioned shortcomings of the approaches (ii) result from the τ -approximation, that is, the reduction of third-order correlations of \mathbf{u} or \mathbf{b} to second-order ones. An important reason for them seems to be an improper treatment of a term connected with the forcing term in the momentum balance.

A proposal is made for a new approach which avoids any conflict with approach (i) (section 5). In this context also a new formalism for the higher-order correlation approximation is presented (section 5.1).

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Appendix A: The limits $q \rightarrow \infty$ and $q \rightarrow 0$

The relations (25) for a_{ip} and b_{ipq} can be written in the form

$$\begin{aligned}
 a_{ip} &= \frac{\epsilon_{ijn}\delta_{lp} - \epsilon_{ijp}\delta_{ln}}{2(4\pi)^{3/2}} q^{5/2} \frac{\tau_c}{\lambda_c} \int_{\infty}^{\infty} \int_0^{\infty} \frac{\exp(-q\xi'^2/4\tau')}{\tau'^{5/2}} Q_{jn}(\mathbf{x}, t; \xi'\lambda_c, -\tau'\tau_c) \xi'_l d^3\xi' d\tau' \\
 &= \frac{\epsilon_{ijn}\delta_{lp} - \epsilon_{ijp}\delta_{ln}}{(4\pi)^{3/2}} q^{3/2} \frac{\tau_c}{\lambda_c} \int_{\infty}^{\infty} \int_0^{\infty} \frac{\exp(-q\xi'^2/4\tau')}{\tau'^{3/2}} \frac{\partial Q_{jn}(\mathbf{x}, t; \xi'\lambda_c, -\tau'\tau_c)}{\partial \xi'_l} d^3\xi' d\tau' \\
 b_{ipq} &= \frac{\epsilon_{ijn}\delta_{lp} - \epsilon_{ijp}\delta_{ln}}{2(4\pi)^{3/2}} \\
 &\quad q^{5/2} \tau_c \int_{\infty}^{\infty} \int_0^{\infty} \frac{\exp(-q\xi'^2/4\tau')}{\tau'^{5/2}} Q_{jn}(\mathbf{x}, t; \xi'\lambda_c, -\tau'\tau_c) \xi'_l \xi'_q d^3\xi' d\tau'.
 \end{aligned} \tag{A1}$$

In order to evaluate (A1) in the limit $q \rightarrow \infty$, it is useful to introduce the new integration variable $\xi'' = \sqrt{q}\xi'$. Then q occurs at no other places in these relations than in the third argument of Q_{jn} , which takes the form $\xi''\lambda_c/\sqrt{q}$. Note that $\xi''\lambda_c/\sqrt{q} = \xi''\sqrt{\eta\tau_c}$. We may have $q \rightarrow \infty$ as a consequence of $\eta\tau_c \rightarrow 0$ or of $\lambda_c \rightarrow \infty$. In the first case, this third argument tends to zero as $q \rightarrow \infty$. In the second case, which seems rather academic, Q_{jn} loses its dependence on this argument as $q \rightarrow \infty$, and we may replace the latter simply by zero. Considering then

$$\int_{\infty} \frac{\exp(-\xi^2/4\tau)}{\tau^{3/2}} d^3\xi = (4\pi)^{3/2}, \quad \int_{\infty} \frac{\exp(-\xi^2/4\tau)}{\tau^{5/2}} \xi_l \xi_q d^3\xi = 2(4\pi)^{3/2} \delta_{lq} \quad (A2)$$

we find (29).

In view of the limit $q \rightarrow 0$, we introduce $\tau' = \tau'/q$. Then q occurs only in the fourth argument of Q_{jn} , that is, $\tau'q\tau_c = \tau''\lambda_c^2/\eta$. We may have $q \rightarrow 0$ due to $\lambda_c^2/\eta \rightarrow 0$ or due to $\tau_c \rightarrow \infty$. In the first case, the fourth argument tends to zero as $q \rightarrow 0$. In the second one, Q_{jn} loses its dependence on this argument as $q \rightarrow 0$, and we may replace it by zero. With

$$\int_0^{\infty} \frac{\exp(-\xi^2/4\tau)}{\tau^{3/2}} d\tau = \frac{2\sqrt{\pi}}{\xi}, \quad \int_0^{\infty} \frac{\exp(-\xi^2/4\tau)}{\tau^{5/2}} d\tau = \frac{4\sqrt{\pi}}{\xi^3} \quad (A3)$$

we then find (30).

Appendix B: An alternative representation of α and β as given by (32)

Let us consider a homogeneous statistically steady turbulence. In this case, the correlation tensor Q_{ij} defined by (23) depends no longer on \mathbf{x} and t , that is $Q_{ij} = Q_{ij}(\xi, \tau)$. We introduce the quantity U_{ij} by

$$U_{ij} = \int_0^{\infty} Q_{ij}(\mathbf{0}, -\tau) d\tau. \quad (B1)$$

With the Fourier transform \hat{Q}_{ij} of Q_{ij} it holds

$$U_{ij} = \int \int_0^{\infty} \hat{Q}_{ij}(\mathbf{k}, -\tau) d^3k d\tau. \quad (B2)$$

Assume now that \hat{Q}_{ij} allows a separation ansatz of the form

$$\hat{Q}_{ij}(\mathbf{k}, \tau) = \hat{Q}_{ij}(\mathbf{k}, 0) f(k, \tau), \quad (B3)$$

where, of course, $f(k, 0) = 1$. Then we have

$$U_{ij} = \int \hat{Q}_{ij}(\mathbf{k}, 0) \hat{\tau}_c(k) d^3k = (2\pi)^{-3} \int Q_{ij}(\xi, 0) \tau_c(\xi) d^3\xi, \quad (B4)$$

where $\hat{\tau}_c(k)$ is defined by

$$\hat{\tau}_c(k) = \int_0^\infty f(k, -\tau) d\tau, \quad (\text{B5})$$

and $\tau_c(\xi)$ such that $\hat{\tau}_c$ is its Fourier transform,

$$\tau_c(\xi) = \int \hat{\tau}_c(k) \exp(i\mathbf{k} \cdot \boldsymbol{\xi}) d^3k. \quad (\text{B6})$$

Comparing (B1) and (B4) and remembering (23) we then arrive at the remarkable relation

$$\int_0^\infty \langle u_i(\mathbf{x}, t) u_j(\mathbf{x}, t - \tau) \rangle d\tau = (2\pi)^{-3} \int \langle u_i(\mathbf{x}, t) u_j(\mathbf{x} + \boldsymbol{\xi}, t) \rangle \tau_c(\boldsymbol{\xi}) d^3\xi. \quad (\text{B7})$$

We restrict our attention now to an incompressible fluid and assume the turbulence to be isotropic. As is well known the Fourier transform \hat{Q}_{ij} of Q_{ij} has then the form

$$\hat{Q}_{ij}(\mathbf{k}, \tau) = \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) g(k, \tau) + i \epsilon_{ijk} \frac{k_k}{k^2} h(k, \tau). \quad (\text{B8})$$

For the calculation of β according to (31) it is sufficient to consider a mirror-symmetric turbulence, that is $h=0$. Then \hat{Q}_{ij} satisfies the assumption (B3). Consequently (B7) applies, and β as given by (32) can also be written in the form

$$\beta = \frac{1}{3(2\pi)^3} \int \langle \mathbf{u}(\mathbf{x}, t) \cdot \mathbf{u}(\mathbf{x} + \boldsymbol{\xi}, t) \rangle \tau_c(\boldsymbol{\xi}) d^3\xi. \quad (\text{B9})$$

We may repeat the above derivations with

$$\tilde{Q}_{ij}(\boldsymbol{\xi}, \tau) = \epsilon_{jkl} \frac{\partial Q_{il}(\boldsymbol{\xi}, \tau)}{\partial \xi_k} \quad (\text{B10})$$

instead of $Q_{ij}(\boldsymbol{\xi}, \tau)$. Using again (B8) we then have

$$\hat{\tilde{Q}}_{ij}(\mathbf{k}, \tau) = \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) h(k, \tau) + i \epsilon_{ijk} k_k g(k, \tau). \quad (\text{B11})$$

For the calculation of α it seems to be justified to put here $g=0$. Then (B11) satisfies the analogue to (B3), and we find an analogue to (B7) with u_j replaced with $(\nabla \times \mathbf{u})_j$. With this result α as given by (32) can be written in the form

$$\alpha = -\frac{1}{3(2\pi)^3} \int \langle \mathbf{u}(\mathbf{x}, t) \cdot (\nabla \times \mathbf{u}(\mathbf{x} + \boldsymbol{\xi}, t)) \rangle \tau_c(\boldsymbol{\xi}) d^3\xi. \quad (\text{B12})$$

The quantities τ_c in (B9) and (B12) may well be different from each other.

Appendix C: $\Omega \times \mathbf{J}$ -effect

Under the assumptions which justify equation (8) we have

$$b_{ijk} = \beta \epsilon_{ijk} - \delta_1 \delta_{ij} \Omega_k - \delta_2 \delta_{ik} \Omega_j - \delta_3 \delta_{jk} \Omega_i. \tag{C1}$$

The β , δ_1 and δ_2 terms lead just to the last line of (8) whereas the δ_3 term has, because of $\nabla \cdot \bar{\mathbf{B}} = 0$, no counterpart there. Multiplying both sides of (C1) first by $\delta_{ij} \Omega_k$ and then by $\delta_{ik} \Omega_j$ and by $\delta_{jk} \Omega_i$ we arrive at a system of three equations for δ_1 , δ_2 and δ_3 . Its solutions for δ_1 and δ_2 read

$$\delta_1 = -\frac{1}{10\Omega^2} (4b_{ijj} - b_{iji} - b_{jii}) \Omega_j, \quad \delta_2 = \frac{1}{10\Omega^2} (b_{ijj} - 4b_{iji} + b_{jii}) \Omega_j. \tag{C2}$$

Expressing b_{ijk} according to (25) we see that δ_1 depends only on the part of Q_{ij} which is firstly antisymmetric in the indices and secondly even in ξ . Assuming that the turbulence is homogeneous and statistically steady, which implies $Q_{ij}(\xi, \tau) = Q_{ji}(-\xi, -\tau)$, we may conclude that this part is odd in τ and therefore vanishes at $\tau = 0$. As for δ_2 this statement applies only under the additional assumption of an incompressible fluid, that is, $\partial Q_{ij} / \partial \xi_j = 0$.

These conclusions can also be drawn immediately from the first results on the $\Omega \times \mathbf{J}$ -effect given by Rädler (1969). By the reason explained above, the functions h and k used there are odd but l is even in τ , and it can be easily shown that l vanishes in the incompressible case. This just confirms the above statements on the part of Q_{ij} responsible for the coefficients that correspond to δ_1 and δ_2 .

By the way, the reason why the $\Omega \times \mathbf{J}$ -effect did not occur in the first calculation of the mean electromotive force \mathcal{E} delivered by Steenbeck *et al.* (1966) was a not completely correct ansatz for Q_{ij} , just without a part being odd in τ .

Appendix D: Derivation of equation (68)

We start from (67) and express v_i and w_j on the right-hand side by their Fourier representations,

$$\Phi_{ij}(\mathbf{v}, \mathbf{w}; \mathbf{x}, \xi) = \int \int \langle \hat{v}_i(\mathbf{k}') \hat{w}_j(\mathbf{k}'') \rangle \exp(i(\mathbf{k}' \cdot (\mathbf{x} + \xi/2) + \mathbf{k}'' \cdot (\mathbf{x} - \xi/2))) d^3 k' d^3 k''. \tag{D1}$$

Introducing new integration variables \mathbf{k} and \mathbf{K} by $\mathbf{k}' = \mathbf{k} + \mathbf{K}/2$ and $\mathbf{k}'' = -\mathbf{k} + \mathbf{K}/2$ we then find

$$\begin{aligned} \Phi_{ij}(\mathbf{v}, \mathbf{w}; \mathbf{x}, \xi) &= \int \hat{\Phi}_{ij}(\mathbf{v}, \mathbf{w}; \mathbf{x}, \mathbf{k}) \exp(i\mathbf{k} \cdot \xi) d^3 k \\ \hat{\Phi}_{ij}(\mathbf{v}, \mathbf{w}; \mathbf{x}, \mathbf{k}) &= \int \langle \hat{v}_i(\mathbf{k} + \mathbf{K}/2) \hat{w}_j(-\mathbf{k} + \mathbf{K}/2) \rangle \exp(i\mathbf{K} \cdot \mathbf{x}) d^3 K. \end{aligned} \tag{D2}$$

The last line is identical to (68).

Appendix E: Higher-order correlation approximations

Consider again an infinitely extended homogeneous fluid and assume for the sake of simplicity $\bar{\mathbf{U}} = \mathbf{0}$ and $\nabla \cdot \mathbf{u} = 0$. Assume in view of the calculation of \mathcal{E} at $\mathbf{x} = \tilde{\mathbf{x}}$ that

$$\bar{B}_i = B_i + B_{ij}(x_j - \tilde{x}_j), \quad (\text{E1})$$

with B_i and B_{ij} independent of position and steady, and $B_{ii} = 0$. Clearly B_i and B_{ij} are equal to \bar{B}_i and $\partial \bar{B}_i / \partial x_j$, respectively, at $\mathbf{x} = \tilde{\mathbf{x}}$. From (4) and (5) it follows that

$$\partial_t b_k - \eta \Delta b_k = (B_p + B_{pq}(x_q - \tilde{x}_q)) \frac{\partial u_k}{\partial x_p} - B_{kp} u_p + (\delta_{kq} \delta_{pr} - \delta_{kr} \delta_{pq}) \frac{\partial}{\partial x_p} (u_q b_r)', \quad \frac{\partial b_i}{\partial x_i} = 0, \quad (\text{E2})$$

and, if again the influence of the initial \mathbf{b} is ignored, that is $t_0 \rightarrow -\infty$,

$$\begin{aligned} b_k(\mathbf{x}, t) &= B_p \int_0^\infty \int_\infty^\infty \frac{\partial G^{(n)}(\xi, \tau)}{\partial \xi_p} u_k(\mathbf{x} - \xi, t - \tau) d^3 \xi d\tau \\ &\quad - B_{pq} \int_0^\infty \int_\infty^\infty \left(G^{(n)}(\xi, \tau) \delta_{kp} \delta_{qr} - \frac{\partial G^{(n)}(\xi, \tau)}{\partial \xi_p} (x_q - \tilde{x}_q - \xi_q) \delta_{kr} \right) u_r(\mathbf{x} - \xi, t - \tau) d^3 \xi d\tau \\ &\quad + (\delta_{kq} \delta_{pr} - \delta_{kr} \delta_{pq}) \int_0^\infty \int_\infty^\infty \frac{\partial G^{(n)}(\xi, \tau)}{\partial \xi_p} (u_q(\mathbf{x} - \xi, t - \tau) b_r(\mathbf{x} - \xi, t - \tau))' d^3 \xi d\tau. \end{aligned} \quad (\text{E3})$$

Then put

$$b_k = b_k^{(1)} + b_k^{(2)} + b_k^{(3)} + \dots \quad (\text{E4})$$

and obtain

$$\begin{aligned} b_k^{(1)}(\mathbf{x}, t) &= B_p \int_0^\infty \int_\infty^\infty \frac{\partial G^{(n)}(\xi, \tau)}{\partial \xi_p} u_k(\mathbf{x} - \xi, t - \tau) d^3 \xi d\tau \\ &\quad - B_{pq} \int_0^\infty \int_\infty^\infty \left(G^{(n)}(\xi, \tau) \delta_{kp} \delta_{qr} - \frac{\partial G^{(n)}(\xi, \tau)}{\partial \xi_p} (x_q - \tilde{x}_q - \xi_q) \delta_{kr} \right) u_r(\mathbf{x} - \xi, t - \tau) d^3 \xi d\tau \end{aligned} \quad (\text{E5})$$

$$b_k^{(\nu)}(\mathbf{x}, t) = (\delta_{kq} \delta_{pr} - \delta_{kr} \delta_{pq}) \int_0^\infty \int_\infty^\infty \frac{\partial G^{(n)}(\xi, \tau)}{\partial \xi_p} (u_q(\mathbf{x} - \xi, t - \tau) b_r^{(\nu-1)}(\mathbf{x} - \xi, t - \tau))' d^3 \xi d\tau, \quad \nu \geq 2.$$

These relations are equivalent to

$$\begin{aligned} b_k^{(\nu)}(\mathbf{x}, t) &= B_p \int_0^\infty \int_\infty^\infty \frac{\partial G^{(n)}(\xi, \tau)}{\partial \xi_p} (v_k^{(\nu)}(\mathbf{x}, t; -\xi, -\tau))' d^3 \xi d\tau \\ &\quad + B_{pq} \int_0^\infty \int_\infty^\infty G^{(n)}(\xi, \tau) (w_{kpq}^{(\nu)}(\mathbf{x}, t; -\xi, -\tau))' d^3 \xi d\tau, \quad \nu \geq 1, \end{aligned} \quad (\text{E6})$$

with

$$\begin{aligned}
 v_k^{(1)}(\mathbf{x}, t; \boldsymbol{\xi}, \tau) &= u_k(\mathbf{x} + \boldsymbol{\xi}, t + \tau), \\
 v_k^{(\nu)}(\mathbf{x}, t; \boldsymbol{\xi}, \tau) &= \int_0^\infty \int_\infty^\infty \frac{\partial G^{(\eta)}(\boldsymbol{\xi}', \tau')}{\partial \xi'_l} \left(u_k(\mathbf{x} - \boldsymbol{\xi}', \tau - \tau') v_l^{(\nu-1)}(\mathbf{x} - \boldsymbol{\xi}', t - \tau'; \boldsymbol{\xi}, \tau) \right. \\
 &\quad \left. - u_l(\mathbf{x} - \boldsymbol{\xi}', \tau - \tau') v_k^{(\nu-1)}(\mathbf{x} - \boldsymbol{\xi}', t - \tau'; \boldsymbol{\xi}, \tau) \right) d^3 \xi' d\tau', \quad \nu \geq 2,
 \end{aligned} \tag{E7}$$

and

$$\begin{aligned}
 w_{kpq}^{(1)}(\mathbf{x}, t; \boldsymbol{\xi}, \tau) &= -\delta_{kp} u_q(\mathbf{x} + \boldsymbol{\xi}, t + \tau) + (x_q - \tilde{x}_q + \xi_q) \frac{\partial u_k(\mathbf{x} + \boldsymbol{\xi}, t + \tau)}{\partial \xi_p} \\
 w_k^{(\nu)}(\mathbf{x}, t; \boldsymbol{\xi}, \tau) &= \int_0^\infty \int_\infty^\infty \frac{\partial G^{(\eta)}(\boldsymbol{\xi}', \tau')}{\partial \xi'_l} \left(u_k(\mathbf{x} - \boldsymbol{\xi}', \tau - \tau') w_{lpq}^{(\nu-1)}(\mathbf{x} - \boldsymbol{\xi}', t - \tau'; \boldsymbol{\xi}, \tau) \right. \\
 &\quad \left. - u_l(\mathbf{x} - \boldsymbol{\xi}', \tau - \tau') w_{kpq}^{(\nu-1)}(\mathbf{x} - \boldsymbol{\xi}', t - \tau'; \boldsymbol{\xi}, \tau) \right) d^3 \xi' d\tau', \quad \nu \geq 2.
 \end{aligned} \tag{E8}$$

From (E4) and (E6) it can be concluded that

$$\begin{aligned}
 b_k(\mathbf{x}, t) &= B_p \int_0^\infty \int_\infty^\infty \frac{\partial G^{(\eta)}(\boldsymbol{\xi}, \tau)}{\partial \xi_p} (v_k(\mathbf{x}, t; -\boldsymbol{\xi}, -\tau))' d^3 \xi d\tau \\
 &\quad + B_{pq} \int_0^\infty \int_\infty^\infty G^{(\eta)}(\boldsymbol{\xi}, \tau) (w_{kpq}(\mathbf{x}, t; -\boldsymbol{\xi}, -\tau))' d^3 \xi d\tau,
 \end{aligned} \tag{E9}$$

where

$$\begin{aligned}
 v_k(\mathbf{x}, t; \boldsymbol{\xi}, \tau) &= v_k^{(1)}(\mathbf{x}, t; \boldsymbol{\xi}, \tau) + v_k^{(2)}(\mathbf{x}, t; \boldsymbol{\xi}, \tau) + v_k^{(3)}(\mathbf{x}, t; \boldsymbol{\xi}, \tau) + \dots \\
 w_{kpq}(\mathbf{x}, t; \boldsymbol{\xi}, \tau) &= w_{kpq}^{(1)}(\mathbf{x}, t; \boldsymbol{\xi}, \tau) + w_{kpq}^{(2)}(\mathbf{x}, t; \boldsymbol{\xi}, \tau) + w_{kpq}^{(3)}(\mathbf{x}, t; \boldsymbol{\xi}, \tau) + \dots,
 \end{aligned} \tag{E10}$$

with $v_k^{(\nu)}$ and $w_{kpq}^{(\nu)}$ as given by (E7) and (E8).

Calculating then $\mathcal{E}_i = \epsilon_{ijk} \langle u_j b_k \rangle$ and expanding it according to (6) we finally find

$$\begin{aligned}
 a_{ip}(\tilde{\mathbf{x}}, t) &= \epsilon_{ijk} \int_0^\infty \int_\infty^\infty \frac{\partial G^{(\eta)}(\boldsymbol{\xi}, \tau)}{\partial \xi_p} \overline{u_j(\tilde{\mathbf{x}}, t) v_k(\tilde{\mathbf{x}}, t; -\boldsymbol{\xi}, -\tau)} d^3 \xi d\tau \\
 b_{ipq}(\tilde{\mathbf{x}}, t) &= \epsilon_{ijk} \int_0^\infty \int_\infty^\infty G^{(\eta)}(\boldsymbol{\xi}, \tau) \overline{u_j(\tilde{\mathbf{x}}, t) w_{kpq}(\tilde{\mathbf{x}}, t; -\boldsymbol{\xi}, -\tau)} d^3 \xi d\tau,
 \end{aligned} \tag{E11}$$

which has to be completed by (E7), (E8) and (E10).

Note added in Proof

In appendix B it has been shown that the results (32) of approach (i) for α and β , with integrals of $\langle \mathbf{u}(\mathbf{x}, t) \cdot (\nabla \times \mathbf{u}(\mathbf{x}, t - \tau)) \rangle$ and $\langle \mathbf{u}(\mathbf{x}, t) \cdot \mathbf{u}(\mathbf{x}, t - \tau) \rangle$ over τ , can be rewritten

into (B9) and (B12), with integrals containing \mathbf{u} only as $\langle \mathbf{u}(\mathbf{x}, t) \cdot (\nabla \times \mathbf{u}(\mathbf{x} + \boldsymbol{\xi}, t)) \rangle$ or $\langle \mathbf{u}(\mathbf{x}, t) \cdot \mathbf{u}(\mathbf{x} + \boldsymbol{\xi}, t) \rangle$, taken over all $\boldsymbol{\xi}$. We have argued there with very specific properties of the turbulence, that is of \mathbf{u} .

Relations between the two types of such integrals, over τ and over $\boldsymbol{\xi}$, can easily be established for a wider range of assumptions. Consider any function $F = F(\boldsymbol{\xi}, \tau)$. Assume that the integral $\int_0^\infty F(\mathbf{0}, -\tau) d\tau$ has a non-zero finite value. Assume further that the integral $\int_\infty F(\boldsymbol{\xi}, 0) g(\boldsymbol{\xi}) d^3\xi$, with a weight function g , has a non-zero finite value, too. Clearly the magnitude of a given g can be fixed such that

$$\int_0^\infty F(\mathbf{0}, -\tau) d\tau = (2\pi)^{-3} \int_\infty F(\boldsymbol{\xi}, 0) g(\boldsymbol{\xi}) d^3\xi \quad (*)$$

There is, of course, some freedom in the choice of the dependence of g on $\boldsymbol{\xi}$. We may determine a specific g , e.g., after the pattern of appendix B. For this purpose we introduce the Fourier transform \hat{F} of F with respect to $\boldsymbol{\xi}$ and put $\int_0^\infty \hat{F}(\mathbf{k}, -\tau) d\tau = \hat{F}(\mathbf{k}, 0) \hat{h}(-\mathbf{k})$. Integrating both sides of this relation over all \mathbf{k} and using the convolution theorem we arrive just at (*), with $g(\boldsymbol{\xi})$ such that its Fourier transform is equal to $\hat{h}(\mathbf{k})$.

Let us first identify F in relation (*) with $\langle \mathbf{u}(\mathbf{x}, t) \cdot (\nabla \times \mathbf{u}(\mathbf{x} + \boldsymbol{\xi}, t - \tau)) \rangle$ or $\langle \mathbf{u}(\mathbf{x}, t) \cdot \mathbf{u}(\mathbf{x} + \boldsymbol{\xi}, t - \tau) \rangle$. Then we arrive without utilizing specific properties of the turbulence at relations which allow us to rewrite the results (32) for α and β into (B9) and (B12).

One could try to rewrite the more general results (29) for a_{ip} and b_{ipq} in an analogous way into relations of type (85), but then two serious difficulties arise, even when ignoring the terms with the factor $(\nu - \eta)$ there. Firstly, due to the admitted inhomogeneity of the turbulence, g will depend on x , too, in contrast to τ_* depending only on $\boldsymbol{\xi}$. Secondly, both a_{ip} and b_{ipq} in (85) contain types of correlations which do not occur in (29). Therefore, the finding explained here extenuates our statement made in section 4.2.2 on the discrepancy between the results of approach (i) and those of approach (iib) at most in the case of homogeneous isotropic turbulence and $q \rightarrow \infty$. But even then it is hard to believe that both integrals in (32) with correlations of \mathbf{u} can be rewritten as integrals over $\boldsymbol{\xi}$ with one and the same g , which could then be related to the τ_* that occurs in (85).