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CAN A DISC DYNAMO WORK IN THE LABORATORY?

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The paper deals with the disc dynamo as often referred to in introductions to dynamo theory. Conditions under which a device of this kind can indeed work as a self–exciting dynamo are derived and discussed with a view to its realization in the laboratory. In any case the radius and the rotation rate of the disc have to be rather large. An estimate using very optimistic assumptions concerning the mechanical stability of the rotating disc and the electric resistance of the relevant parts of the device shows that, as long as no magnetizable material is used, a self-exciting dynamo might be possible, e.g., with a radius of 0.6 m and a rotation rate of 40 s^{-1}. A proper arrangement of magnetizable material clearly improves the conditions for the dynamo.

1. Introduction. In many introductions to the theory of cosmic dynamos their principle is explained with reference to the so-called disc dynamo. Although a dynamo of this kind seems very simple, to our best knowledge it has never been realized in a laboratory. In this paper we discuss the conditions which are necessary for a disc dynamo to work.

We consider a device as depicted in Fig. 1. A metallic disc, rigidly connected with a metallic axis, rotates in the presence of a magnetic field about this axis. Two non–moving sliding contacts (brushes), one at the rim of the disc and the other at the axis, are electrically connected by a wire which is wound to form a coil. Due to the rotation of the disc an electric potential difference is created between rim and axis, which drives an electric current through the coil and thus produces a magnetic field. If the coil is properly connected this field amplifies the original one. Under certain conditions, which should be explained in what follows, an initially given magnetic field can be amplified or maintained without any external causes which otherwise would lead to a magnetic field. In this case the device works as a self–exciting dynamo.

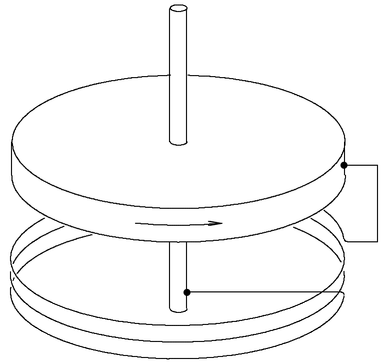

Fig. 1. The device considered.

2. General considerations.

2.1. For the electric circuit constituted by the coil and its connections to rim and axis of the disc as well as by disc and axis themselves we have

$$L\frac{\mathrm{d}I}{\mathrm{d}t} + RI = \nu\phi\,. \tag{1}$$

Here I is the electric current in this circuit, L the inductance and R the resistance of the circuit, ν the rotation rate of the disc and ϕ the magnetic flux that penetrates it. For this flux we write

$$\phi = L'I\,, \tag{2}$$

with a coefficient L', which can be interpreted as mutual inductance between the considered circuit and a hypothetical one coinciding with the rim of the disc. We will later relate L' to one of the contributions to L.

From (1) and (2) we conclude that

$$L\frac{\mathrm{d}I}{\mathrm{d}t} + (R - \nu L')I = 0 \tag{3}$$

and, assuming that L, L', R and ν are independent of I,

$$I(t) = I(0)\exp(pt) \tag{4}$$

with

$$p = \frac{\nu L' - R}{L}\,. \tag{5}$$

The device considered can work as a self-exciting dynamo as soon as the condition $p \geq 0$ is satisfied. We write it in the form

$$\nu \geq \nu_{\mathrm{crit}} \tag{6}$$

with a critical rotation rate ν_{crit}, given by

$$\nu_{\mathrm{crit}} = R/L'\,. \tag{7}$$

For the growth time T defined by $pT = 1$ we have

$$T = \frac{L}{\nu L' - R}\,. \tag{8}$$

2.2. We distinguish between two parts of the electric circuit, the coil and the remaining part.

As for the inductance L we assume that due to the geometry of the device it can be considered as a sum of two contributions,

$$L = L_{\mathrm{c}} + L_{\mathrm{r}}\,, \tag{9}$$

where L_{c} is the inductance of the coil and L_{r} that of the remaining part of the circuit.

The resistance R can in any case be understood as the sum $R_{\mathrm{c}} + R_{\mathrm{r}}$, where R_{c} is the resistance of the coil and R_{r} that of the remaining parts of the circuit including the sliding contacts. We note that the resistance of such contacts is in general only for small currents independent of them. In a wide range of larger currents the voltage occurring there (the brush voltage) is almost independent of them, that is, the resistance decreases with growing currents. Since we are only interested in the onset of a dynamo at small I we consider the contact resistance and so R_{r} and R as independent of I. In view of what follows we write

$$R = R_{\mathrm{c}}(1+\beta)\,, \quad \beta = R_{\mathrm{r}}/R_{\mathrm{c}}\,. \tag{10}$$

For the coefficient L' we put

$$L' = \frac{\gamma}{n}L_{\mathrm{c}}\,, \tag{11}$$

where n means the number of turns of the coil and γ a factor between 0 and 1. In the ideal case in which all magnetic flux through the coil penetrates the disc this factor is equal to 1.

The critical rotation rate $\nu_{\rm crit}$ defined by (7) can now be written in the form

$$\nu_{\rm crit} = \zeta\nu_0\,, \quad \zeta = \frac{1+\beta}{\gamma}\,, \quad \nu_0 = \frac{nR_{\rm c}}{L_{\rm c}}\,. \tag{12}$$

Here ν_0 is the ideal critical rotation rate, that is, the value of $\nu_{\rm crit}$ for the ideal case $\zeta = 1$, or $\beta = 0$ and $\gamma = 1$. Clearly $\nu_{\rm crit}$ is independent of $L_{\rm r}$. Only the growth time T is influenced by the latter.

The case in which the resistance of the sliding contacts and therefore $R_{\rm r}$ depend on I is outside the scope of this paper. We only point out that in this case, that is, for not too small I, the rotation rate which is necessary for dynamo action might be below that calculated for the limit of small I. If then a dynamo works with some finite I at a rotation rate below that for this limit, I should permanently grow until, e.g., the braking effect of the Lorentz force on the disc stops this process.

3. The case of a cylindrical coil.

3.1. We first consider a cylindrical coil with an rectangular cross-sectional area of winding, and we assume that it does not contain any magnetizable material. Its inductance $L_{\rm c}$ is given by

$$L_{\rm c} = \frac{\mu_0}{4\pi} n^2 r f(\Delta r/r, l/r)\,; \tag{13}$$

see e.g. [1] or [2]. Here μ_0 means the magnetic permeability of free space and n again the number of turns. Further r is the mean radius and Δr the thickness of the winding (so that $r + \Delta r/2$ is its outer and $r - \Delta r/2$ its inner radius) and l its length. The function f can be represented with the help of elliptic integrals. A few characteristic values of f are given in Table 1. There and in what follows we use the notations $\Delta r/r = \rho$ and $l/r = \lambda$.

The resistance $R_{\rm c}$ of the coil is given by

$$R_{\rm c} = n\frac{2\pi r}{\sigma q}\,, \tag{14}$$

where σ means the electric conductivity of the material of the wire and q its cross section.

3.2. For the ideal critical rotation rate ν_0 introduced with (12) we find, when using (13) and (14),

$$\nu_0 = \frac{8\pi^2}{\mu_0 \sigma q f(\rho, \lambda)}\,. \tag{15}$$

Table 1. Some values of f (derived from relations and data given in [1]).

	$\lambda = 0$	0.1	0.2	0.5	1.0	2.0	5.0	10.0
$\rho = 0$	∞	48.9	40.1	28.9	20.8	13.6	6.71	3.63
0.1	48.9	40.3	34.9	26.3	19.6	12.9	6.42	3.50
0.2	40.1	34.7	31.4	24.4	18.4	12.4	6.17	3.38
0.5	29.7	27.1	25.1	20.1	15.8	10.8	5.48	3.01
1.0	21.0	19.6	18.3	15.5	12.7	8.78	4.48	2.47
2.0	12.5	12.0	11.6	10.6	9.21	7.03	3.02	1.74

Interestingly enough, the number n of turns does not occur here. Furthermore, the dimensions of the device, that is r, l and Δr, occur only in the form of the ratios ρ and λ. This suggests that the dynamo can work, at least in the ideal case defined by $R_\mathrm{r} = 0$ and $\gamma = 1$, in arbitrarily small dimensions. However, we have to require that $\Delta r \cdot l > nq$ or, what is the same, that the filling factor ξ, defined by

$$\xi = \frac{nq}{\Delta r\, l}\,, \tag{16}$$

is smaller than unity. By this reason certain small dimensions have to be excluded.

For the following discussion we write relation (15) with the help of (16) in the form

$$\nu_0 = \frac{8\pi^2 n}{\mu_0 \sigma r^2 \xi g(\rho,\lambda)}\,, \quad g(\rho,\lambda) = \rho\lambda f(\rho,\lambda)\,. \tag{17}$$

If in addition to $\mu_0\sigma$ also r, n and ξ are fixed, ν_0 decreases with growing g. As Table 2 shows, g grows with both ρ and λ. In order to have ν_0 small, both ρ and λ should be large.

For an example we put, thinking of a copper coil, $\sigma = 5.9 \cdot 10^7\,\mathrm{Sm}^{-1}$, that is, $\mu_0\sigma = 74.1\,\mathrm{m}^{-2}\mathrm{s}$, and choose in addition $r = 0.1\,\mathrm{m}$, $\xi = 0.95$ and $\rho = \lambda = 1$. Then we have

$$\nu_0 = 8.83\, n\, \mathrm{s}^{-1}\,. \tag{18}$$

If ζ would be close to unity, with a not too large number n of turns we could hope to arrive at feasible values of the critical rotation rate ν_crit.

3.3. In order to find an estimate for the parameter ζ let us again consider the resistance R_c of the coil. From (14) and (16) we have

$$R_\mathrm{c} = \frac{2\pi n^2}{\sigma\xi\rho\lambda r}\,. \tag{19}$$

Using (10) and (12) we arrive then at

$$\zeta = \frac{1}{\gamma}\left(1 + \frac{\sigma\xi\rho\lambda r R_\mathrm{r}}{2\pi n^2}\right)\,. \tag{20}$$

With the above values of σ, r, ξ, ρ and λ we have $R_\mathrm{c} = 1.12 \cdot 10^{-6}\, n^2\,\Omega$. Accepting the additional assumptions $\gamma = 0.9$ and $R_\mathrm{r} = 10^{-2}\,\Omega$, the realization of which seems to be possible, we find further

$$\zeta = 1.11\left(1 + \frac{8.92 \cdot 10^3}{n^2}\right)\,. \tag{21}$$

At least for small numbers n of turns rather large values of ζ are to be expected. That is, we arrive under all circumstances at large values of ν_crit, whose realization

Table 2. Some values of g.

	$\lambda = 0$	0.1	0.2	0.5	1.0	2.0	5.0	10.0
$\rho = 0$	0	0	0	0	0	0	0	0
0.1	0	0.40	0.70	1.33	1.96	2.58	3.21	3.50
0.2	0	0.69	1.26	2.44	3.68	4.96	6.17	6.67
0.5	0	1.36	2.51	5.03	7.90	10.8	13.7	15.1
1.0	0	1.96	3.66	7.75	12.7	17.6	22.4	24.7
2.0	0	2.50	4.82	10.6	18.4	28.1	30.2	34.8

will prove to be problematic.

3.4. We now focus our attention on the critical rotation rate ν_{crit} defined by (12), take ν_0 from (17), further ζ from (20), and find

$$\nu_{\mathrm{crit}} = \frac{8\pi^2 n}{\mu_0 \sigma r^2 \xi \gamma g}\left(1 + \frac{\sigma\xi\rho\lambda r R_{\mathrm{r}}}{2\pi n^2}\right). \tag{22}$$

Clearly ν_{crit}, considered as function of n, possesses just one minimum. Denoting the corresponding value of n by n_{opt} we have

$$n_{\mathrm{opt}} = \sqrt{\frac{\sigma\xi\rho\lambda r R_{\mathrm{r}}}{2\pi}}\,. \tag{23}$$

At $n = n_{\mathrm{opt}}$ we have $\beta = 1$. There the critical rotation rate ν_{crit} takes the value $\nu_{\mathrm{crit\,min}}$ given by

$$\nu_{\mathrm{crit\,min}} = \frac{8\pi}{\mu_0\gamma}\sqrt{\frac{2\pi R_{\mathrm{r}}}{\sigma\xi r^3}}\,h(\rho,\lambda)\,, \quad h(\rho,\lambda) = \frac{\sqrt{\rho\lambda}}{g(\rho,\lambda)}\,. \tag{24}$$

Some values of h are shown in Table 3. They indicate that there is a flat minimum of h at the largest possible value of ρ, that is $\rho = 2$, and a value of λ close to 2.

Using again the above values of σ, r, ξ, ρ, λ, R_{r} and γ we find

$$n_{\mathrm{opt}} = 94.4\,, \quad \nu_{\mathrm{crit\,min}} = 1.86 \cdot 10^3\,\mathrm{s}^{-1}\,. \tag{25}$$

This implies a velocity of $1170\,\mathrm{ms}^{-1}$ at the rim of the disc. As we will see in the following this is far beyond the bounds for the mechanical stability of the disc.

3.5. Since the rotating disc is subject to centrifugal forces the velocity at the rim, $2\pi\nu r$, has to be below a certain bound, say v_{limit}. We identify here ν with $\nu_{\mathrm{crit\,min}}$, that is, consider the condition

$$2\pi\nu_{\mathrm{crit\,min}} r < v_{\mathrm{limit}}\,. \tag{26}$$

Accepting again the above values of σ, ξ, γ, ρ and λ but leaving r and R_{r} free we write this in the form

$$v_{\mathrm{limit}}^2 r/R_{\mathrm{r}} > 1.36 \cdot 10^7\,\mathrm{m}^3\mathrm{s}^{-2}\Omega^{-1}\,. \tag{27}$$

For a simple copper disc v_{limit} is at best equal to $30\,\mathrm{ms}^{-1}$, for an aluminum disc about $60\,\mathrm{ms}^{-1}$. Higher values of v_{limit} require special materials and technical measures. Let us optimistically put $v_{\mathrm{limit}} = 150\,\mathrm{ms}^{-1}$. If we then remain at $R_{\mathrm{r}} = 10^{-2}\,\Omega$ a dynamo can work only with $r > 6\,\mathrm{m}$, which implies $\nu < 4\,\mathrm{s}^{-1}$. With $R_{\mathrm{r}} = 10^{-3}\,\Omega$, only $r > 0.6\,\mathrm{m}$ has to be required, and we have $\nu < 40\,\mathrm{s}^{-1}$. The resistance R_{r} could perhaps be lowered to a value of these order of magnitude by using several symmetrically arranged sliding contacts instead of a single one at

Table 3. Some values of h.

	$\lambda = 0$	0.1	0.2	0.5	1.0	2.0	5.0	10.0
$\rho = 0$	∞	∞	∞	∞	∞	∞	∞	∞
0.1	∞	0.25	0.20	0.17	0.16	0.17	0.22	0.29
0.2	∞	0.20	0.16	0.13	0.12	0.13	0.16	0.21
0.5	∞	0.17	0.13	0.10	0.090	0.093	0.12	0.15
1.0	∞	0.16	0.12	0.091	0.079	0.081	0.10	0.13
2.0	∞	0.19	0.14	0.094	0.077	0.071	0.10	0.13

rim and axis of the rotating disc. In this case each contact at the rim could be connected with a separate wire of the coil now consisting of several wires which come then again together at the axis. A smaller device, for example with $r = 0.1\,\mathrm{m}$ as assumed above, can hardly work as a dynamo.

3.6. Let us give an estimate of the potential difference U between rim and axis of the rotating disc and the corresponding current I through the coil. For this purpose we assume that the dynamo works in a steady state with a magnetic flux density B defined by $B = \phi/\pi r^2$. Then we have

$$U = \nu\phi = \pi\nu B r^2\,, \quad I = U/(R_\mathrm{c} + R_\mathrm{r})\,. \tag{28}$$

Let us, thinking of the last example, choose $\nu = 40\,\mathrm{s}^{-1}$, $r = 0.6\,\mathrm{m}$ and $R_\mathrm{c} = R_\mathrm{r} = 10^{-3}\,\Omega$. If then, for example, $B = 10^{-2}\,\mathrm{T}$ it follows that $U = 0.45\,\mathrm{V}$ and $I = 230\,\mathrm{A}$. In this case the electric power converted into heat inside the dynamo is about 100 W. At least this power is needed to maintain the rotation of the disc. Of course, this example has to be considered with caution since in particular the above assumption of small I might be no longer correct.

4. The case of a single turn.

4.1. The considerations of the preceding section should also apply to the case in which there is a single turn only, that is $n = 1$, instead of a coil. In order to demonstrate this in an independent way we consider now a thin torus with circular cross–section instead of the coil. We imagine that this torus is slotted at one place and that the two ends occurring in this way are electrically connected with the rim and the axis of the disc.

The inductance L_c of a torus is given by

$$L_\mathrm{c} = \mu_0 r(\ln(r/r_\mathrm{w}) + c)\,; \tag{29}$$

see e.g. [1], [2] or [3]. As above μ_0 means the magnetic permeability of free space, r the large and r_w the small radius of the torus. Relation (29) is an approximation for $r_\mathrm{w}/r \ll 1$, obtained from a more general one containing elliptic integrals. Finally c means a positive constant not exceeding 0.33, whose exact value depends on the kind of that approximation but is meaningless in the limit $r_\mathrm{w}/r \to 0$. In analogy to (13) we write (29) in the form

$$L_\mathrm{c} = \frac{\mu_0}{4\pi} r f'(r_\mathrm{w}/r)\,, \quad f'(r_\mathrm{w}/r) = 4\pi(\ln(r/r_\mathrm{w}) + c)\,. \tag{30}$$

A few values of f', calculated with $c = 0.33$, are shown in Table 4. There and in what follows the notation $\rho' = r_\mathrm{w}/r$ is used.

The resistance R_c of the torus with $\rho' \ll 1$ follows from (14) if we put there $n = 1$, that is,

$$R_\mathrm{c} = \frac{2\pi r}{\sigma q}\,. \tag{31}$$

4.2. Analogously to (15) we write the relation for the ideal critical rotation rate ν_0 following from (12), (30) and (31) in the form

$$\nu_0 = \frac{8\pi^2}{\mu_0 \sigma q f'(\rho')}\,. \tag{32}$$

Using

$$q = \pi r_\mathrm{w}^2 \tag{33}$$

we further write (32) in a form corresponding to (17),

$$\nu_0 = \frac{8\pi^2}{\mu_0 \sigma r^2 g'(\rho')}\,, \quad g'(\rho') = \pi {\rho'}^2 f'(\rho')\,. \tag{34}$$

Table 4. Some values of f' and g', calculated with $c = 0.33$.

ρ'	0	0.05	0.1	0.2	0.3	0.4	0.5
$f'(\rho')$	∞	41.8	33.1	24.4	19.3	15.7	12.9
$g'(\rho')$	0	0.326	1.036	3.267	5.458	7.896	10.13

The function g' vanishes at $\rho' = 0$, first grows with ρ', reaches a maximum at $\rho' = \exp(-1/2 + c)$, e.g. for $c = 0.33$ at $\rho' = 0.844$, and then decays again. A few values of g', again calculated with $c = 0.33$, are given in Table 4. Since our considerations apply to small ρ' only, the maximum mentioned is outside the region of interest.

Let us compare the functions $f(\rho, \lambda)$ and $g(\rho, \lambda)$ introduced for the coil, in the case of a quadratic cross–section, $\rho = \lambda$, with the functions $f'(\rho')$ and $g'(\rho')$ for the torus under the assumption of equal cross–sections, $\rho' = \rho/\sqrt{\pi}$. For $\rho' < 0.1$ the functions f and f' as well as g and g' essentially agree with each other. The relative deviations do not exceed 15 % if all $c < 0.5$ are considered and are even below 1 % for $c = 0.33$. That is, the results of the preceding section indeed apply in some approximation to $n = 1$, too.

5. Concluding remarks. The realization of a disc dynamo as considered so far seems to be possible only with rather large disc radii. For a particular estimate we have made two very optimistic assumptions. Firstly, we have admitted velocities of the rim of the disc up to $150\,\mathrm{ms}^{-1}$. Secondly, we considered it as possible to minimize the electric resistance for the current through axis, disc and sliding contacts so that it takes a value of only about $10^{-3}\,\Omega$. Using these and a few other, rather natural assumptions we have then shown that a dynamo can only work if the disc radius exceeds 0.6 m.

Having in mind that the disc dynamo should demonstrate the principle of cosmic dynamos we considered here a device without any magnetizable material. A properly arranged part of magnetically highly permeable material as, for example, soft iron could considerably enlarge the magnetic flux through the rotating disc. In this way the requirements for the rotation rate of the disc could presumably be lowered by at least a factor 10. Then the self–excitation condition could be met with some technical efforts even in a smaller device.

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