In the Forschungszentrum Karlsruhe an experiment has been built up which demonstrates a homogeneous dynamo similar to that supposed to be acting in the Earth’s interior. This experiment has been discussed within the framework of mean-field dynamo theory. In a preceding paper [5] kinematic aspects of this approach to the experiment have been explained. The present paper deals with the nonlinear regime of the dynamo, that is, the regime determined by the back-reaction of the magnetic field on the fluid motion. Using again the mean-field concept a simple theory describing the behaviour of the dynamo in this regime is developed. On this basis predictions are made on the dynamic behaviour of the dynamo and in particular on the magnitude of the magnetic field and the rates of the fluid flows in the saturated states. They are compared with the experimental results.

1. Introduction. In the Forschungszentrum Karlsruhe a device has been constructed for an experiment demonstrating a homogeneous dynamo as is assumed to exist in the Earth’s interior or in cosmic bodies [7]. This experiment has been run the first time successfully in December 1999 [2, 3, 8, 9]. In a preceding paper [5] we have explained the basic ideas of a mean-field theory of this experiment, in which an $\alpha$-effect plays a crucial part, and outlined its elaboration as far as kinematic aspects are concerned. In particular, the self-excitation conditions for magnetic fields in dependence on the rates of the fluid flows have been derived. Here we extend this theory to the nonlinear regime of the dynamo, that is, the regime essentially determined by the back-reaction of the magnetic field on the fluid flows. Other contributions to the understanding of this nonlinear regime have been delivered by Busse and Tilgner [10, 11].

In Section 2 we explain our model of the dynamo in the nonlinear regime in its general form. In Section 3 it is utilized to estimate the magnitudes of the magnetic field and the rates of the fluid flows that occur in the absence of any externally imposed magnetic field and without any influence of the magnetic field on the $\alpha$-effect coefficient other than that via the flow rates. In Section 4 our model is simplified by replacing the mean-field induction equation by two ordinary differential equations for two scalar quantities characterizing the mean magnetic field. Using this simplified model we discuss in Section 5, permitting an externally imposed magnetic field and a more general dependence of the $\alpha$-effect coefficient on the magnetic field, the relations between the magnetic field and the rates of the fluid flows or the parameters characterizing the pressures of the pumps driving these flows. In Section 6 we summarize our results and the conclusions concerning the situation in the experiment.

We take the preceding paper mentioned [5] as the basis for the present one and call it paper I. Its sections or appendices X, equations (Y), figures or tables Z are referred to as I.X, (I.Y) or I.Z. We use the notation introduced there.
2. A model of the dynamo in the nonlinear regime.

2.1. General description of the model. A complete theory of the dynamo should be based on the induction equation for the magnetic field and the Navier–Stokes equation, the continuity equation etc. for the fluid flow taking into account the Lorentz force exerted by the magnetic field on the fluid. This force will in general act against the flows generated by the pumps and thus limit the magnitude of the magnetic field. However, the detailed application of the mentioned equations to the experimental device is a very complex task. We hope to comprehend essential features of the dynamo by a simple form of the mean-field induction equation combined with two ordinary differential equations for the rates of flow through the axial and helical channels of the dynamo module.

Let us start with the induction equation for the mean magnetic field \( \mathbf{B} \). As in Section I.3 we consider no other induction effect than the anisotropic \( \alpha \)-effect. Thus inside dynamo module we have

\[
\partial_t \mathbf{B} = - \nabla \times \left( \eta \nabla \times \mathbf{B} + \alpha_\perp (V_C, V_H, \mathbf{B}) (\mathbf{B} - (\mathbf{e} \cdot \mathbf{B}) \mathbf{e}) \right), \quad \nabla \cdot \mathbf{B} = 0. \tag{1}
\]

Here \( \eta \) is again the magnetic diffusivity of the fluid and \( \mathbf{e} \) the unit vector parallel to the axis of the dynamo module. Concerning the dependence of the coefficient \( \alpha_\perp \) on the flow rates \( V_C \) and \( V_H \) we refer to Section I.4 and note that we will give equations below which relate \( V_C \) and \( V_H \) via the magnetic pressures in the axial and helical channels to \( \mathbf{B} \). However, the change of \( \alpha_\perp \) with the magnetic field is in general not completely described by the changes of \( V_C \) and \( V_H \). There may be in addition a change of \( \alpha_\perp \) connected with the influence of the magnetic field on the profiles of the fluid flows in the channels or by motions caused by the magnetic field occur in regions which in the original concept contain stagnant fluid only; see [4, 10]. The argument \( \mathbf{B} \) of \( \alpha_\perp \) should remind of these possibilities. In this sense we will speak in the following of “quenching via the flow rates” and “quenching via the flow profiles”, respectively. In this context we recall a result of Section I.4.3 according to which \( \alpha_\perp \) for a class of simple flow profiles obeying (I.23), if calculated in the second-order approximation, depends only on the flow rates \( V_C \) and \( V_H \) and not on the flow profiles. In the following we use again the dimensionless measure \( C \) defined by (I.12) for the \( \alpha \)-effect, with \( \alpha_\perp \) and \( \eta \) interpreted as values at the center of the dynamo module and \( R \) the being radius of this module, and we denote the marginal values of \( C \) again by \( C^* \).

Of course, the equations (1) have to be completed by proper boundary conditions. In the simplest case we think of conditions that ensure the continuation of the field \( \mathbf{B} \) inside the dynamo module as a potential field in outer space that vanishes at infinity. We will, however, also admit a continuation as another field of external origin which simulates either the field that unavoidably exists in the laboratory or a field intentionally generated by Helmholtz coils arranged outside the dynamo module.

We point out that there is one circuit in the experimental device which contains the central channels of all 52 spin generators but there are two circuits for the helical channels, each feeding 26 of them. Here these two circuits are assumed to be equal to each other, that is, are described by one and the same flow rate \( V_H \).

The behavior of the flow rates \( V_C \) and \( V_H \) is governed by the Navier–Stokes equation and the continuity equation applied to the loops with the central and the helical channels. They imply the balance of the forces exerted by the pumps with the inertial forces, the pressure and frictional forces and the Lorentz forces. As explained in more detail in Appendix A we formulate this balance in the form

\[
dt V_C = \kappa_C \left( P_C(V_C) - R_C(V_C) - L_C(V_C, \mathbf{B}) \right)
\]

\[
dt V_H = \kappa_H \left( P_H(V_H) - R_H(V_H) - L_H(V_C, V_H, \mathbf{B}) \right). \tag{2}
\]
$P_C$ and $P_H$ are the pressures generated by the pumps in these circuits, $R_C$ and $R_H$ the pressure drops due to the hydraulic resistances, $L_C$ and $L_H$ the pressure drops due to the Lorentz force. These quantities will be specified in the following two subsections. The coefficients $\kappa_C$ and $\kappa_H$ are given by

$$\kappa_C = \frac{\gamma_C s_C}{\rho C}, \quad \kappa_H = \frac{\gamma_H s_H}{\rho H}.$$  

Here $\gamma_C$ and $\gamma_H$ are factors defined more precisely below, which are close to unity and depend slightly on the flow profiles, $s_C$ and $s_H$ mean the cross-sections of the central and helical channels, $\rho$ the mass density of the fluid, and $l_C$ and $l_H$ the total lengths of the loops with central and helical channels outside the pumps. Whereas the definitions of $s_C$ and $l_C$ are obvious, those of $s_H$ and $l_H$ may need some explanation. We measure $l_H$ along the central line of the loop and define then $s_H$ such that $s_H l_H$ is just the volume of the loop.

We represent the fluid velocities $u_C$ and $u_H$ in the loops with the central and helical channels in the form

$$u_C = \frac{V_C}{s_C} \tilde{u}_C, \quad u_H = \frac{V_H}{s_H} \tilde{u}_H,$$

where $\tilde{u}_C$ and $\tilde{u}_H$ are steady dimensionless vector fields. Note that (4) implies a separation of space and time dependences since $V_C$ and $V_H$ depend only on the time, and $\tilde{u}_C$ and $\tilde{u}_H$ only on the position. As shown in Appendix A, $\gamma_C^{-1}$ and $\gamma_H^{-1}$ are just equal to the averages of $\tilde{u}_C^2$ and $\tilde{u}_H^2$ over the volumes of the loops with the central and helical channels. If rigid-body motion in these channels is assumed and the connections between the channels and to the pumps are ignored we have $\gamma_C = 1$ and $\gamma_H = 0.9015$.

2.2. The characteristics of the pumps and the hydraulic resistances. For the pressures generated by the pumps we refer to Appendix C and put

$$P_C = k_C P_C^0 (1 - c_{PC} V_C), \quad P_H = k_H P_H^0 (1 - c_{PH} V_H).$$

The factors $k_C$ and $k_H$ indicate with which fractions of the maximally attainable pressures the pumps really work, $0 < k_C, k_H \leq 1$. Further, $P_C^0$ and $P_H^0$ are these maximum pressures, and $c_{PC}$ and $c_{PH}$ are constants that take into account the pressure drops inside the pumps under load.

For the pressure losses due to the hydraulic resistances we assume, referring again to Appendix C,

$$R_C = R_C^0 \left( 1 + c_{RC} \left( 1 + \frac{c_{RC}}{V_C} \right)^{1/4} \right)^{1/2}, \quad R_H = R_H^0 V_H^2,$$

where $R_C^0$, $R_H^0$, $c_{RC}$ and $c_{RC}'$ are constants. The main contributions to the resistances are due to the bends of the tubes.

Numerical values for the parameters that occur in (5) and (6) will be given in Section 2.4.

2.3. The effect of the Lorentz force. Roughly speaking, the pressure drops $L_C$ and $L_H$ in the loops containing the central channels or helical channels are averages of the components of the Lorentz force in flow direction, taken over the volume of those parts of the loops which are penetrated by the magnetic field, multiplied by the lengths of those parts. Indeed the derivations of Appendix A show that

$$L_C = - (\tilde{u}_C \cdot \gamma)_{C} m_C l_C, \quad L_H = - (\tilde{u}_H \cdot \gamma)_{H} m_H l_H,$$
where \( \mathbf{u}_C \) and \( \mathbf{u}_H \) are the dimensionless vectors in flow direction introduced with (4). \( \mathbf{f} \) means the Lorentz force exerted on a unit volume. \( \langle \cdots \rangle_C \) and \( \langle \cdots \rangle_H \) denote averages over the volume of the parts of the loops which are penetrated by the magnetic field, and \( l_C \) and \( l_H \) the lengths of these parts measured along the central stream lines.

Let us specify the velocities \( \mathbf{u}_C \) and \( \mathbf{u}_H \) inside the spin-generators according to (I.23). That is, referring to a proper cylindrical coordinate system such as introduced with respect to the cell \( 0 \leq x, y \leq a \) in Section I.4.3 we have

\[
\mathbf{u}_C = -\frac{u(\varrho)}{u_0} (0, 0, 1), \quad \mathbf{u}_H = -\frac{s_H \omega(\varrho)}{h \Delta \varrho} \omega_0 \left( 0, \frac{\varrho}{\varrho_m}, \frac{h}{2\pi \varrho_m} \right),
\]

where \( u_0 \) and \( \omega_0 \) are the averages of \( u(\varrho) \) and \( \omega(\varrho) \) taken over the sections of the central and helical channels with a plane perpendicular to the axis of the spin-generator, and \( \varrho_m = (\varrho_1 + \varrho_2)/2 \), \( \Delta \varrho = \varrho_2 - \varrho_1 \), with \( \varrho_1 \) and \( \varrho_2 \) being the inner and outer radius of the helical channel, respectively. We ignore the contributions of the connecting flows between the spin-generators to \( \langle \mathbf{u}_C \cdot \mathbf{f} \rangle_C \) and \( \langle \mathbf{u}_H \cdot \mathbf{f} \rangle_H \). Then \( \langle \cdots \rangle_C \) and \( \langle \cdots \rangle_H \) can simply be interpreted as averages over the mentioned sections of the central or the helical channel with a plane perpendicular to the axis of the spin-generator.

When calculating the Lorentz force \( \mathbf{f} \) we assume for the sake of simplicity that \( \mathbf{B} \) is a homogeneous field. Then we have

\[
\mathbf{f} = \frac{1}{\mu} (\nabla \times \mathbf{B}') \times (\mathbf{B} + \mathbf{B}'),
\]

where \( \mu \) is the magnetic permeability of free space and \( \mathbf{B}' \) again the fluctuating part of the magnetic field. We consider a steady situation so that \( \mathbf{B}' \) is governed by the corresponding version of equations (I.13), that is

\[
\eta \nabla^2 \mathbf{B}' + (\mathbf{B}' \cdot \nabla) \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{B}' = -(\mathbf{B} \cdot \nabla) \mathbf{u}, \quad \nabla \cdot \mathbf{B}' = 0,
\]

where \( \mathbf{u} \) is again the fluid velocity.

We may assume that \( \mathbf{B}' \), like \( \mathbf{B} \), is independent of \( z \). Therefore we can put

\[
\mathbf{B}' = \mathbf{e} \times \nabla \Psi + \mathbf{e} B'_z
\]

with \( \Psi \) and \( B'_z \) independent of \( z \), too, and conclude from (10) that

\[
\eta \Delta \Psi - \mathbf{u}_\perp \cdot \nabla \Psi = \mathbf{e} \cdot (\mathbf{u}_\perp \times \mathbf{B}_\perp), \quad \eta \Delta B'_z - \mathbf{u}_\perp \cdot \nabla B'_z = \mathbf{e} \cdot (\nabla (\mathbf{e} \cdot \mathbf{u}_\parallel) \times \nabla \Psi) - \nabla (\mathbf{e} \cdot \mathbf{u}_\parallel) \cdot \mathbf{B}_\perp.
\]

Here \( \mathbf{u}_\perp \) and \( \mathbf{u}_\parallel \) stand for the parts of \( \mathbf{u} \) perpendicular and parallel to \( \mathbf{e} \), respectively. Analogously, \( \mathbf{B}_\perp \) means the part of \( \mathbf{B} \) perpendicular to \( \mathbf{e} \). Note that \( \mathbf{B}' \) does not depend on the other part of \( \mathbf{B} \), that is, on \( B_z \).

For a first approximation we restrict ourselves to the part of \( \mathbf{f} \) which is linear in \( \mathbf{u} \), that is, we ignore contributions to \( \mathbf{f} \) which are of higher order in \( \mathbf{u} \). As can be easily followed up \( \mathbf{B} + \mathbf{B}' \) in (9) has then to be replaced by \( \mathbf{B} \) and, as in the second-order approximation defined for the calculation of \( \alpha_\perp \) (see Appendix I.B), the terms with \( \mathbf{u} \) not containing \( \mathbf{B} \) in (10) and (12) have to be cancelled. In addition, with arguments similar to those used in the context of \( \alpha_\perp \) (and explained in Appendix I.C) we can conclude that for the calculation of \( \langle \mathbf{u}_C \cdot \mathbf{f} \rangle_C \) and \( \langle \mathbf{u}_H \cdot \mathbf{f} \rangle_H \) for a given cell merely the motion in this cell is relevant while that in the neighboring cells can be ignored. So we may take the solution \( \mathbf{B}' \) of (10) given in the Appendix I.B and calculate \( \langle \mathbf{u}_C \cdot \mathbf{f} \rangle_C \) and \( \langle \mathbf{u}_H \cdot \mathbf{f} \rangle_H \) immediately on the basis
of (9). Alternatively, we can proceed as in Appendix B. On both ways we find

\[ L_C = \frac{\sigma V_C^m}{2\gamma_C^m s_C} B_\perp^2 \psi_C(V_C, V_H), \quad L_H = \frac{\sigma V_H^m}{2\gamma_H^m s_H} B_\perp^2 \psi_H(V_C, V_H) \]

(13)

where \( \sigma \) is the electric conductivity of the fluid, \( B_\perp \) the modulus of \( \mathbf{B}_\perp \), and \( \gamma_C^m \) and \( \gamma_H^m \) are averages of \( \tilde{u}_C^2 \) and \( \tilde{u}_H^2 \) over the parts of the loops penetrated by the magnetic field.

Let us now leave our first approximation, that is, the restriction to linearity of \( \mathbf{f} \) in \( \mathbf{u} \). However, as before we ignore in the following the influence of the fluid motion in the neighboring spin-generators on the magnetic field in the considered one. This is not completely correct, but has to be regarded as an approximation which remains to be checked\(^1\). Furthermore, for the sake of simplicity we restrict ourselves to rigid-body motion of the fluid, that is, put \( u/u_0 = 1 \) and \( \omega/\omega_0 = 1 \).

We represent the result, which has been derived as described in Appendix B, in the form

\[ L_C = \frac{\sigma V_C^m}{2\gamma_C^m s_C} B_\perp^2 \psi_C(V_C, V_H), \quad L_H = \frac{\sigma V_H^m}{2\gamma_H^m s_H} B_\perp^2 \psi_H(V_C, V_H) \]

(14)

with two functions \( \psi_C \) and \( \psi_H \) satisfying \( \psi_C(V_C, 0) = 1 \) for \( V_C \neq 0 \) and \( \psi_H(V_C, 0) = 1 \) for all \( V_C \). These functions calculated for \( g_1 = a/4 \) and \( g_2 = a/2 \) are shown in Fig. 1. Here \( \gamma_C^m \) and \( \gamma_H^m \) have to be specified to rigid body motion, that is, in analogy to (36) we have \( \gamma_C^m = 1 \) and \( \gamma_H^m = 0.9015 \).

We complete now equations (2) by

\[ L_C = B_\perp^2 \tilde{L}_C, \quad \tilde{L}_C = c_{LC} V_C \psi_C(V_C, V_H), \quad c_{LC} = \frac{\sigma V_C^m}{2\gamma_C^m s_C}, \]

\[ L_H = B_\perp^2 \tilde{L}_H, \quad \tilde{L}_H = c_{LH} V_H \psi_H(V_C, V_H), \quad c_{LH} = \frac{\sigma V_H^m}{2\gamma_H^m s_H}. \]

(15)

2.4. The numerical parameters of the model. On the basis of (1) and (2), completed by (I.30), (5), (6) and (15), we may calculate the quantities \( V_C, V_H \) and \( \mathbf{B} \) if, for example, \( k_C \) and \( k_H \) are given. For this purpose we need the numerical values of \( k_C, k_H, F_C^0, F_H^0, c_{PC}, c_{PH}, R_C^0, R_H^0, c_{RC}, c'_{RC} \), \( c_{LC} \) and \( c_{LH} \). As explained

\(^1\)see Note added in proof at the end of the paper
The magnetic field and converted into heat by ohmic dissipation. Since some of
This implies \( \Delta^2 \) according to
allows us to determine a pair, or possibly several pairs, of values
\( V \) can afterwards find the corresponding value of
all pairs of \( V \) and \( H \) are meaningless.

### 3. The magnetic field and the flow rates in the absence of an external field.
Let us first consider the model of the dynamo defined in the preceding
nal field. For the sake of simplicity we further assume that \( \alpha \) and therefore \( C \) depends only via the flow rates \( V_C \) and \( V_H \) on the magnetic field, that is, ignore any quenching via the flow profiles. In order to stress this assumption we write here \( C_0 \) instead of \( C \). If then \( \mathbf{B} \), \( V_C \) and \( V_H \) represent a solution of the equations governing our model, the same applies to \( -\mathbf{B} \), \( V_C \) and \( V_H \), too.

We have seen in several examples, for which the equations (1) and (2) have been integrated numerically, that the evolution of \( \mathbf{B} \), \( V_C \) and \( V_H \) ends up in a steady state. Having this in mind we restrict our attention here to steady states only.

Steady solutions of the equations for \( \mathbf{B} \), that is (1), require that \( C_0(V_C, V_H) \) takes its marginal value \( C^* \). Hence the consequences of the equations (1) and (2), for the steady case read simply

\[
C_0(V_C, V_H) = C^*,
\]

\[
P_C(V_C) - R_C(V_C) - B_{\perp}^2 \tilde{L}_C(V_C, V_H) = 0,
\]

\[
P_H(V_H) - R_H(V_H) - B_{\perp}^2 \tilde{L}_H(V_C, V_H) = 0.
\]

Eliminating \( B_{\perp}^2 \) from the last two lines of (17) we find

\[
(P_C(V_C) - R_C(V_C)) \tilde{L}_H(V_C, V_H) - (P_H(V_H) - R_H(V_H)) \tilde{L}_C(V_C, V_H) = 0.
\]

If all other relevant parameters are given the first line of (17) together with (18) allows us to determine a pair, or possibly several pairs, of values \( V_C \) and \( V_H \) without considering \( B_{\perp}^2 \). With the help of the second or the third line of (17) we can afterwards find the corresponding value of \( B_{\perp}^2 \). We have, however, to discard all pairs of \( V_C \) and \( V_H \) for which \( B_{\perp}^2 \) takes negative values.

In Table 1 the flow rates \( V_C \) and \( V_H \) and the quantity \( B_{\perp} \) characterizing the magnitude of the generated magnetic field in steady states of the dynamo are listed for a few values of \( C^* \) close to those confirmed experimentally, and various values of \( k_C \) and \( k_H \), which determine the regime of the pumps. We recall that \( k_C P_C^0 \) and \( k_H P_H^0 \) are the total pressures generated by the pumps in the loops with the central and helical channels, that is, each of them is the sum of the pressure drop inside the pump and that due to the hydraulic resistances and the Lorentz forces inside the other parts of the loop. We denote these pressure drops in the other parts of the loops by \( \Delta PC \) and \( \Delta PH \), that is, \( \Delta PC = RC + LC \) and \( \Delta PH = RH + LH \). This implies \( \Delta PC = k_C P_C^0 (1 - CP_C V_C) \) and \( \Delta PH = k_H P_H^0 (1 - CP_H V_H) \). In Table 1 also the total power \( N \) needed to maintain the steady states is given, calculated according to \( N = \Delta PC V_C + 2 \Delta PH V_H \), as well as the relative fraction \( f_{\text{ohm}} \) fed into the magnetic field and converted into heat by ohmic dissipation. Since some of
The flow rates $V_C$ and $V_H$ and the measure $B_\perp$ of the magnitude of the magnetic field for steady states of the dynamo, further the total power $N$ needed to maintain these states and its relative fraction $f_{\text{ohm}}$ corresponding to ohmic dissipation.

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<th>$k_H$</th>
<th>$V_C$ [m$^3$/h]</th>
<th>$V_H$ [m$^3$/h]</th>
<th>$B_\perp$ [G]</th>
<th>$N$ [kW]</th>
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<tr>
<td></td>
<td>0.5</td>
<td>0.5</td>
<td>126</td>
<td>123</td>
<td>0</td>
<td>24</td>
<td>0</td>
</tr>
</tbody>
</table>

The numerical values in (16) have considerable uncertainties, the values of $B_\perp$, $V_C$, $V_H$, $N$ and $f_{\text{ohm}}$ must be understood as rough estimates. In this sense they are in fair agreement with the experimental results.

In Fig. 2 the dependence of $B_\perp$ on $k_C$ and $k_H$ is represented for a typical value of $C^*$ obtained from the experimental data, namely $C^* = 9$.

4. A simplified model of the dynamo. Let us now proceed to considerations involving an externally imposed magnetic field and a more complex dependence of $\alpha_\perp$ on the magnetic field. For this purpose, however, we simplify our model of the dynamo by replacing the equations (1) for $\vec{B}$, which are partial differential equations, by two ordinary differential equations for two scalar quantities reflecting essential features of $\vec{B}$.

Fig. 2. The dependence of $B_\perp$ on $k_C$ and $k_H$ for $C^* = 9$. The representation applies also with $B_\perp$ replaced by $-B_\perp$. 

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The field $\mathbf{B}$ can always be represented as a sum of a poloidal and a toroidal part, $\mathbf{B} = \mathbf{B}^p + \mathbf{B}^T$, where $\mathbf{B}^p = -\nabla \times (e \times \mathbf{S})$ and $\mathbf{B}^T = -e \times \nabla T$ with defining scalars $S$ and $T$. We assume for our simple consideration that $\eta$ and $\alpha_\perp$ do not depend on the radial and the azimuthal coordinates $r$ and $\varphi$ of the cylindrical system adjusted to the dynamo module. Then we can conclude from (1) that
\begin{align}
\partial_t \mathbf{B}^p &= -\nabla \times (\eta \nabla \times \mathbf{B}^p) + \nabla \times (\alpha_\perp \mathbf{B}^T) \\
\partial_t \mathbf{B}^T &= -\nabla \times (\eta \nabla \times \mathbf{B}^T) + \nabla \times (\alpha_\perp (\mathbf{B}^p - (e \cdot \mathbf{B}^p) e)).
\end{align}
Let us now characterize the fields $\mathbf{B}^p$ and $\mathbf{B}^T$ by the two scalars $B^p$ and $B^T$, respectively. We relate their magnitudes to the magnitudes of $\mathbf{B}^p$ and $\mathbf{B}^T$ and their signs to the directions of $\mathbf{B}^p$ and $\mathbf{B}^T$ so that a change of the signs of $B^p$ and $B^T$ means an inversion of the directions of $\mathbf{B}^p$ and $\mathbf{B}^T$. The externally imposed magnetic field, $\mathbf{B}_0$, has to be poloidal. We characterize it by a scalar $B_0$ and allow analogously to $B^p$ and $B^T$ both signs of $B_0$. The equations (19) for $\mathbf{B}^p$ and $\mathbf{B}^T$ and the explanations concerning $\mathbf{B}_0$ suggest that the behavior of $B^p$ and $B^T$ can be described roughly by
\begin{align}
\partial_t B^p &= -\lambda^p (B^p - B_0 - \zeta^T C B^T) \\
\partial_t B^T &= -\lambda^T (B^T - \zeta^T C B^p).
\end{align}
Here $\lambda^p$ and $\lambda^T$ are positive quantities representing decay rates of the order of $\eta/R^2$, $C$ is again the dimensionless measure of the $\alpha$-effect introduced above and $\zeta^p$ and $\zeta^T$ are numerical factors of the order of unity. We note that in the steady case with $C = 0$ we have simply $B^p = B_0$ and $B^T = 0$.

Consider for a moment the homogeneous version of the equations (20), that is, that with $B_0 = 0$. It has solutions such that both $B^p$ and $B^T$ vary with $t$ like $\exp(\lambda t)$, where
\begin{align}
\lambda &= -\frac{1}{2} (\lambda^p + \lambda^T) \left( 1 \pm \sqrt{1 - \frac{4 \lambda^p \lambda^T}{(\lambda^p + \lambda^T)^2} (1 - \zeta^p \zeta^T C^2)} \right).
\end{align}
For the case of the lower sign a steady solution, $\lambda = 0$, is possible. Since we have then $C = C^*$ it turns out that $\zeta^p \zeta^T C^{*2} = 1$.

Using this result we bring the equations (20) in the form
\begin{align}
\partial_t B^p &= -\lambda^p \left( B^p - C \zeta^T C^* B^T \right) \\
\partial_t B^T &= -\lambda^T \left( B^T - \zeta^{-1} C \zeta^p B^p \right),
\end{align}
where $\zeta$ is a new numerical factor of the order of unity, $\zeta = \sqrt{\zeta^p / \zeta^T}$. The parameters $\lambda^p$, $\lambda^T$ and $\zeta$ can be chosen such that these equations reproduce special features of the models defined by the partial differential equations (1). We refer here to the model described in Section I.6.6 in which $\alpha_\perp$ is no longer constant in the whole dynamo module but varies near its boundary, more precisely to the most easily excitable $\mathbf{B}$-field in this model, which is of type $m = 1$. When requiring that the dependence of $\lambda$ on $C/C^*$ given with Fig. I.20 is approximated by (21) with the lower sign and choosing arbitrarily $\lambda^p < \lambda^T$ we have
\begin{align}
\lambda^p &= 2.06 \text{s}^{-1}, \quad \lambda^T = 30.8 \text{s}^{-1}
\end{align}
The ratio $\lambda^T/\lambda^P$ is surprisingly high. The higher it is the quicker follows the evolution of $B^T$ for a given $C/C^*$ that of $B^P$, in other words, the more is $B^T$ enslaved by $B^P$. The factor $\zeta$ is related to the ratio of $B^P$ and $B^T$ in the steady case. The parameters $\lambda^P$, $\lambda^T$ and $\zeta$ are, however, meaningless for the self-excitation condition of the dynamo and for the determination of $B^P$ in the steady case.

For the following we define our model of the dynamo in the nonlinear regime by the equations (22) together with (2). Clearly, $|B^P|$ and $B_\perp$ have to be of the same order of magnitude. Specifying the definition of $B^P$ further we simply identify $|B^P|$ with $B_\perp$. We confess that we proceed here with some arbitrariness.

We could also identify $B_\perp$, for example, with $|B^T|$ or $\sqrt{B^T|^2 + B^P|^2}$. This would not change the essential features of the results reported in this paper.

A numerical code has been developed which allows of solving the initial value problem for the equations (22) and (2), that is, to follow up the evolution of $B^P$, $B^T$, $V_C$ and $V_H$. The calculations reported in this paper have been carried out with the values (23) for $\lambda^P$ and $\lambda^T$ and with $\zeta = 1$. As mentioned above, however, these values are meaningless for all results on steady states.

5. The magnetic field and the flow rates in the presence of an externally imposed field.

5.1. Steady states: the dependence of the magnetic field on the flow rates.

Let us now evaluate the equations (22) for the steady state of the dynamo. They imply

\[(1 - (C/C^*)^2)B^P - B_0 = 0 \tag{24}\]

and, in addition, $B^T = \zeta^{-1}(C/C^*)B^P$. Using this last relation, the condition (24) can easily be expressed also in terms of $B^T$ where, however, a dependence on $\zeta$ occurs. As a consequence of (2) for the steady state of the dynamo the second and third line of (17) with $B_\perp$ interpreted as $|B^P|$ apply, too.

We discuss, however, first the dependence of $B^P$ on $V_C$ and $V_H$ determined by (24) and consider in this context the last two lines of (17) only as relations from which afterwards, that is, for given $B^P$, $V_C$ and $V_H$, the parameters $k_C$ and $k_H$ defining the regime of the pumps can be calculated. Of course, cases in which $k_C$ or $k_H$ exceeds unity have to be considered as unrealistic.

For a first step we assume that $\alpha_\perp$ and therefore $C$ depends only via the flow rates $V_C$ and $V_H$ on the magnetic field. As above we write then again $C_0$ instead

\[\text{Fig. 3. The dependence of } B^P/B_0 \text{ in steady states on } V_H \text{ for } C^* = 9 \text{ and } V_C = 100 \text{ m}^3/\text{h}. \text{ The solid line describes stable states, the dashed line unstable ones. The dash-dotted line at } V_H = 111.61 \text{ m}^3/\text{h} \text{ separates the region with } C_0 < C^* \text{ from that with } C_0 > C^*. \text{ If, e.g., } B_0 = 1 \text{ G the conditions } k_C, k_H \leq 1 \text{ are only satisfied for } V_H \leq 111.31 \text{ m}^3/\text{h} \text{ and } 111.93 \text{ m}^3/\text{h} \leq V_H \leq 159.74 \text{ m}^3/\text{h}.\]
of $C$. Then relation (24) defines $B^P/B_0$ as a function of $V_C$ and $V_H$,

$$\frac{B^P}{B_0} = \frac{1}{1 - (C_0/C^*)^2}.$$  \hspace{1cm} (25)

Fig. 3 shows an example of the dependence of $B^P/B_0$ on $V_H$ for fixed $C^*$ and $V_C$. In general, that is for arbitrary $C^*$ and $V_C$, the branch of positive $B^P/B_0$ corresponds to $C_0 < C^*$, that with negative $B^P/B_0$ to $C_0 > C^*$. We note that for positive $B^P/B_0$ the external field characterized by $B_0$ supports the regeneration of the toroidal field from the poloidal one, while for negative $B^P/B_0$ it hinders this regeneration. That is why steady states occur in the first case for $C_0 < C^*$, in the second case for $C_0 > C^*$.

The stability of the steady states has been tested by numerical integration of the equations (2) and (22). The states on the first-mentioned branch, that is, the one with positive $B^P/B_0$, prove to be stable, the other ones unstable. Fig. 4 shows isolines of $B^P/B_0$ for stable states, again with a fixed $C^*$, in the $V_CV_H$-plane.

For the next step we assume that $C$ depends no longer only via $V_C$ and $V_H$ on the magnetic field. According to our above explanations we permit an additional quenching via the flow profiles such that

$$C = C_0(V_C, V_H) \, q(|B^P|),$$  \hspace{1cm} (26)

where $q$ is some kind of quenching function. The argument $|B^P|$ of $q$ has been chosen again with some arbitrariness, following the principle of maximum simplicity. Arguments like $|B^T|$ or $\sqrt{B^{P^2} + B^{T^2}}$ seem reasonable, too. In the steady case considered here they differ from $|B^P|$ only by factors depending on $C/\zeta C^*$ and do not lead to basically different results.

For the following we choose, again striving for simplicity,

$$q = \frac{1}{1 + \epsilon B^{P^2}}$$  \hspace{1cm} (27)

with some positive constant $\epsilon$.

We may formulate equations (22) with $B^P/B_0$ and $B^T/B_0$ instead of $B^P$ and $B^T$, and (24) with $B^P/B_0$ instead of $B^P$. Then it is useful to rewrite (27) by putting

$$\epsilon B^{P^2} = \epsilon'(B^P/B_0)^2,$$  \hspace{1cm} (28)

which implies $\epsilon' = \epsilon B_0^2$. In this way it becomes clear that $B^P/B_0$ depends on $\epsilon$ and $B_0$ only via $\epsilon B_0^2$. When thinking of the form of $q$ modified in such a way we...
write \( q(\epsilon', B^P/B_0) \) instead of \( q(B^P) \). As an example we note that in the case of \( \epsilon' = 10^{-5} \) and \( B_0 = 1 \, \text{G} \) we have \( q = 0.91 \) for \( B^P = 100 \, \text{G} \), or \( q = 0.71 \) for \( B^P = 200 \, \text{G} \).

When combining (24) with (26), (27) and (28) we arrive at an equation of the fifth degree in \( B^P/B_0 \). In general, such equations have no unique solution but may have, depending on the values of \( C^* \) and \( \epsilon' \), three or even five solutions. For our purposes it is useful to bring (24) with \( C \) expressed by (26), (27) and (28) in the form

\[
\frac{C_0(V_C, V_H)}{C^*} = \frac{1}{q(\epsilon', B^P/B_0)} \sqrt{1 - \frac{B_0}{B^P}}. \tag{29}
\]

The left-hand side depends only on \( V_C \), \( V_H \) and \( C^* \), the right-hand side only on \( \epsilon' \) and \( B^P/B_0 \). For \( q = 1 \) equation (29) is in accordance with (25). It should be noted that there are no steady solutions of (22) with \( 0 \leq B^P/B_0 < 1 \), and thus (29) does not apply to these values of \( B^P/B_0 \). We understand (29) as a relation that defines, for instance, \( V_H \) for given \( V_C \), \( C^* \) and \( \epsilon' \) as a function of \( B^P/B_0 \). This function proves to be unique. The connection between \( V_H \) and \( B^P/B_0 \) has been determined for various \( V_C \), \( C^* \) and \( \epsilon' \) by solving (29) iteratively. Examples of the results are shown in Fig. 5. As will become clear later, the values for \( \epsilon' \) chosen there are of the same order of magnitude as those determined from experimental results. In general we have \( C < C^* \) for the branch with positive \( B^P/B_0 \), and \( C > C^* \) for the branches with negative \( B^P/B_0 \). Using the code which solves the initial value problem for the equations (22) combined with (2), with the numerical values given in (16) and (23) we have tested the stability of the steady states. It turned out that the states on the branch with positive \( B^P/B_0 \) are always stable. If there are two states with negative \( B^P/B_0 \) for a given \( V_H \) then that with the largest modulus of \( B^P/B_0 \) is stable, too, but the other one is unstable. Analogously to Fig. 5, in Fig. 6 isolines of \( B^P/B_0 \) in the \( V_C/V_H \)-plane are shown for stable states.

We point out that many of the results presented in this section do neither depend on the parameters \( \lambda^P \), \( \lambda^V \) and \( \zeta \) nor on the equations (2) for \( V_C \) and \( V_H \) and the parameters entering there, which are specified in (16). These parameters

\[
\text{Fig. 5. } B^P/B_0 \text{ versus } V_H \text{ for steady states with } C^* = 9, V_C = 100 \, \text{m}^3/\text{h}, \text{ and various values of } \epsilon', \text{ left } \epsilon' = 10^{-7}, \text{ right } \epsilon' = 10^{-5}. \text{ The solid lines describe stable states, the dashed line describes unstable ones. The dash-dotted line corresponds to } C_0 = C^*, \text{ that is, } V_H = 111.61 \, \text{m}^3/\text{h}. \text{ For the branches with positive or negative } B^P/B_0 \text{ applies } C < C^* \text{ or } C > C^*, \text{ respectively. If, e.g., } B_0 = 1 \, \text{G}, \text{ in the case } \epsilon' = 10^{-7} \text{ the condition } kC, k_H \leq 1 \text{ is satisfied for the branch with positive } B^P/B_0 \text{ if } V_H \leq 112.24 \, \text{m}^3/\text{h}, \text{ for that with negative } B^P/B_0 \text{ and unstable or stable states if } V_H \leq 159.73 \, \text{m}^3/\text{h} \text{ or } V_H \leq 112.86 \, \text{m}^3/\text{h}, \text{ respectively, in the case } \epsilon' = 10^{-5} \text{ analogously if } V_H \leq 144.91 \, \text{m}^3/\text{h}, \text{ } V_H \leq 159.74 \, \text{m}^3/\text{h} \text{ or } V_H \leq 145.39 \, \text{m}^3/\text{h}.\]
and equations have been used only to obtain the statements on the stability of the steady states and on the coefficients $k_C$ and $k_H$.

5.2. Comparison with the experimental data. In the experiment measurements of components of the magnetic field in the center of the dynamo module have been carried out for fixed $V_C$ but variable $V_H$ [3, 9]. The diagrams obtained in this way (e.g., Fig. 5 in [3] and [9]) agree in their essential features with those of our Fig. 5 rather than with Fig. 3. We note that the diagrams derived from the measured data show components of the local field but these differ only by some factor from the components of the mean field (see Appendix I.D). The agreement of the diagrams mentioned with those of our Fig. 5 strongly suggests that the additional quenching via the flow profiles, described by $q$, plays an important part in the experiment. That is, the back-reaction of the magnetic field on the fluid motion is more complex than a simple reduction of the flow rates $V_C$ and $V_H$ (see also [4]).

It is of interest to determine from the magnetic fields, measured for example for a given $V_C$ and various $V_H$, the critical value of $V_H$ defined such that at this value in the absence of an externally imposed field a very small magnetic field would neither grow nor decay or, in other words, such that it just satisfies $C_0(V_C, V_H) = C^*$. For the determination of this critical value of $V_H$ two different procedures, both of empirical nature, were used.

In the first case a diagram of the kind discussed above with a curve representing a component of the measured magnetic field and resembling the upper curves in the diagrams of Fig. 5 has been considered, and the turning point of this curve was assigned to the critical value of $V_H$ (Müller and Stieglitz, private communication). Provided there is no noticeable difference between the shapes of the curves for the measured local magnetic field and the mean field, at least in the situations assumed for Fig. 5 the critical value of $V_H$ is slightly underestimated by this procedure.

In the second case the measured pressure drops $\Delta p_C$ and $\Delta p_H$ in the loops with central and helical channels have been considered. It was for example observed that the curve representing $\Delta p_H$ over $V_H$, which has been derived from the experimental data, shows a clear change of its slope at some value of $V_H$, and this value was taken as the critical value [3, 9]. In order to check this way of determination of the critical $V_H$ we have calculated $\Delta p_C$ and $\Delta p_H$ for some examples. Two results are represented in Fig. 7. Note that in the absence of the magnetic field $\Delta p_C$ is expected to be independent of $V_H$, and $\Delta p_H$ to show a quadratic dependence on $V_H$. This can be seen easily when considering the left parts of the curves for the stable states with positive $B^P/B_0$ and the right parts of the curves for the unstable
Fig. 7. The pressure drops $\Delta p_C$ and $\Delta p_H$ in the loops with the central and with helical channels of the spin-generators versus $V_H$ in an example with $C^*$ and $V_C$ as in Fig. 5, $B_0 = 1$ G and various $\epsilon'$, again left $\epsilon' = 10^{-7}$ and right $\epsilon' = 10^{-5}$. Meaning of line styles as in Fig. 5.

states with negative $B_P^*/B_0$. In both examples the slope of the curve for $\Delta p_C$ is for small $V_H$ close to zero but increases strongly as soon as $V_H$ exceeds a certain value. We may define this value more precisely by approximating the two parts of such a curve in its neighbourhood by two straight lines. We see, however, that the value determined in this way is slightly below the critical value of $V_H$ which we are looking for. The points in which the curves for $\Delta p_H$ deviate from a parabola defined by small $V_H$ can be clearly recognized in the case $\epsilon' = 10^{-7}$ but less clearly in the case $\epsilon' = 10^{-5}$. But in these points $V_H$ is again slightly below the wanted critical value of $V_H$.

We have adjusted bifurcation diagrams as represented in Fig. 5, which are defined by the values of $C^*$, $V_C$ and $\epsilon'$, to several sets of measured data (Müller and Stieglitz, private communication). Let $B^{exp}$ be a component of the magnetic field measured in the center of the dynamo module for a given $V_C$ and various values of $V_H$. We expect that it can be represented in the form $B^{exp} = B_c(B^P/B_0)$ where $B_c$ is a constant with the dimension of a magnetic field and $B^P/B_0$ is understood in the above sense as a function of $C^*$, $V_C$, $V_H$ and $\epsilon'$. We may indeed choose $B_c$, $C^*$ and $\epsilon'$ so that the measured data agree in some approximation with the calculated data. An example is shown in Fig. 8. In this case, in which $B^{exp}$ means the $y$-component of the magnetic field in the sense of Fig. 1.1, we have striven to approximate the branch with positive $B^{exp}$ as well as possible. Under the condition $V_C = 112.5 \text{m}^3/\text{h}$ met in the experiment this is reached with $C^* = 9.189$, $B_c = 0.285$ G and $\epsilon' = 1.47 \times 10^{-7}$. In all examples investigated we
observed that the values of $\epsilon'$ are of the order of $10^{-7}$ or even smaller. We point out the fact that if some agreement of the calculated branch with positive $B^\text{exp}$ and the experimental data is achieved there is a systematic deviation of the branch with negative $B^\text{exp}$ from these data. This analogously applies also to the example investigated by Tilgner and Busse [10]. We interpret this fact as an indication of an asymmetry of the experimental device with respect to the plane $y = 0$.

We note that $B^\text{exp}$, if considered in the above sense as a function of $B_c, C^*, V_C, V_H$ and $\epsilon'$, is independent of $B_0$. We assume, however, that $B_c$ is of the order of $B_0$, that is, $B_c = \chi B_0$ with a numerical factor $\chi$ of the order of unity, which depends on the ratio of the considered component to the total field and on the relation between the local and the mean field in the center of the dynamo module (see Appendix I.D).

In contrast to $B^P/B_0$ or $B^\text{exp}$, the pressure drops $\Delta p_C$ and $\Delta p_H$, as shown in Fig. 7, depend on $B_0$. In particular, the slopes of the curves for $\Delta p_C$ and $\Delta p_H$ in the regions in which $V_H$ exceeds its critical value become larger with growing $B_0$. We have calculated $\Delta p_C$ and $\Delta p_H$ as functions of $V_H$ for the above example with $V_C = 112.5 \text{ m}^3/\text{h}$, $C^* = 9.189$ and $\epsilon' = 1.47 \times 10^{-7}$ for various $B_0$. Fig. 9 shows results for $B_0 = 0.25 \text{ G}$. In this case the slopes of the mentioned curves agree fairly with those of the curves representing the measured data. We therefore may take this value of $B_0$ as realistic. In this way we find $\chi = 1.14$. If the mean

\[ \Delta p_C (\text{MPa}) \]
\[ \Delta p_H (\text{MPa}) \]

Fig. 9. The pressure drops $\Delta p_C$ and $\Delta p_H$ (solid and dashed lines used as in Fig. 5) in the loops with the central and the helical channels of the spin-generators versus $V_H$ for the values of $C^*, V_C$ and $\epsilon'$ given with Fig. 8 and $B_0 = 0.25 \text{ G}$ in comparison with measured data (squares and triangles; the squares of the right part of the figure correspond to one of the two loops with helical channels, the triangles to the other one).
field at the axis of the dynamo module were aligned with the y-axis we had to expect $\chi = 1 + \epsilon_y$ with $\epsilon_y = 0.91$ (see Appendix I.D), that is, $\chi = 1.91$. The discrepancy between these two values of $\chi$ can partially be explained by assuming that there is no such alignment. The fact that the measured pressure differences $\Delta p_C$ are always clearly smaller than the calculated ones indicates that the value of $R_C^0$ given in (16) and used in all our calculations is somewhat too high. The systematic difference in the values of $\Delta p_H$ measured in the two loops with helical channels has again to be interpreted as a consequence of an asymmetry of the experimental device.

5.3. Steady states: the dependence of the magnetic field on the pressures generated by the pumps. So far we have considered the magnetic field, characterized by $B^P$, in steady states in its dependence on the flow rates $V_C$ and $V_H$. Let us now proceed to another view and study as in Section 3 the magnetic field and the flow rates, again in steady states, as functions of the parameters $k_C$ and $k_H$ characterizing the regime of the pumps. Concerning the interpretation of $k_C$ and $k_H$ and their connections with the pressure drops $\Delta p_C$ and $\Delta p_H$ due to the hydraulic resistances and the Lorentz forces we refer to Section 3.

In order to determine steady states in that sense we may start again from relation (24), replace there $C$ according to (26) by $C_0q$ and express the arguments $V_C$ and $V_H$ of $C_0$ according to the last two lines of (17), completed by (5), (6) and (15), by $k_C$, $k_H$ and $B^P$. In Fig. 10 the dependence of $C_0(B^P)/C_0(0)$ on $B^P$, which occurs in this way, is shown for several values of $k_C$ and $k_H$ and compared
with that of \( q(B^P) \) on \( B^P \). For moderate \( B^P \) the quenching via the flow rates is of the same order as that via the flow profiles, for larger \( B^P \) stronger.

We recall our results concerning the dependence of \( B^P \), or \( B_\perp \), on \( k_C \) and \( k_H \) presented in Table 1 and Fig. 2. They have now to be characterized more precisely by \( B_0 = 0 \) and \( \epsilon = 0 \). Analogous results have been derived for stable steady states with finite \( B_0 \). First, again \( \epsilon = 0 \) was assumed so that there is no additional quenching via the flow profiles. Figs. 11 and 12 show the dependence of \( B^P \) for a given \( B_0 \) on \( k_C \) and \( k_H \).

The case with non-zero \( \epsilon \), that is, with additional quenching via the flow rates, has been investigated, too. The results are similar to those represented in Figs. 11 and 12. Of course, the additional quenching implies some reduction of the magnitudes of the magnetic field.

6. Conclusions.

6.1. Non-steady cases: the evolution of the magnetic field and the flow rates.

Finally we give some examples for the evolution of the magnetic field and the flow rates. In the left part of Fig. 13 such an evolution is shown for a case with \( \epsilon = 0 \) and finite \( B_0 \) starting from a state with vanishing \( B^P \) and \( B^T \) but finite \( V_C \) and

Fig. 12. The dependence of \( B^P \) for stable steady states with \( C^* = 9, B_0 = 1 \text{G} \) and \( \epsilon = 0 \) on \( k_C \) and \( k_H \).

Fig. 13. Examples of the evolution of \( B^P, B^T, V_C \) and \( V_H \) with different initial conditions, for \( C^* = 9, k_C = k_H = 0.7, B_0 = 1 \text{G} \) and \( \epsilon = 0 \). Note that values of \( \lambda^P \) and \( \lambda^T \) according to (23) and \( \zeta = 1 \) were assumed.
It ends up in a steady state with positive $B^P$ and $B^T$. The situation depicted in the right part of Fig. 13 differs from that in the left one by starting from a state in which $B^P$ and $B^T$ are negative. Here, the evolution leads to a final steady state with negative $B^P$ and $B^T$. In cases with non-zero $\epsilon$ very similar evolutions have been observed.

Within the framework of the mean-field theory of the Karlsruhe dynamo a model describing its behaviour in the nonlinear regime has been developed, which considers also an externally imposed magnetic field and various kinds of quenching of the $\alpha$-effect coefficient $\alpha_\perp$, or the corresponding dimensionless quantity $C$. In addition to the quenching resulting from the dependence of the fluid flow rates $V_C$ and $V_H$ on the magnetic field also that resulting from the influence of the magnetic field on the flow profiles is considered.

Ignoring the external field and the quenching via the flow profiles we have given estimates of the magnitude of the magnetic field and the flow rates in steady states of the dynamo, which are in fair agreement with the measured data.

We also studied the dependence of the magnitude of the magnetic field on the flow rates in the presence of an external field for cases without and with additional quenching via the flow profiles. In the last case each of the diagrams showing the magnitude of the magnetic field for fixed $V_C$ in its dependence on $V_H$ exhibits two branches corresponding to stable states of the dynamo, one existing for all $V_H$ and another one not connected with the former and existing only for values of $V_H$ which exceed a certain bound. The directions of the magnetic fields in the states on the two branches are opposite to each other. Diagrams of that kind agree in their essential features with such derived from the measured data. This strongly suggests that the additional quenching, that is, the influence of the magnetic field on the profiles of the flows is important for the real dynamo.

We have shown that the dependences of the magnetic field and of the pressure drops in the fluid circuits on the flow rate $V_H$, if calculated with properly chosen parameters, are in fair agreement with the measured data. The remaining discrepancies are presumably due to asymmetries of the experimental device which were not considered in our theory.

In this context we checked also procedures used for the determination of the self-excitation threshold of the dynamo from the measured data. All examples considered suggest that these procedures slightly underestimate the self-excitation threshold.

We further have presented results concerning the dependence of the magnitude of the magnetic field on the pressure drops in the fluid circuits.

Finally examples of the evolution of the magnetic fields and the flow rates have been given, which demonstrate how steady final states can be reached that differ in the direction of the magnetic field.

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Appendix A. On the relations between the Lorentz force and the magnetic pressure drops $L_C$ and $L_H$. We assume that the fluid velocity $u$ in the loops with the spin generators satisfies the momentum balance and the continuity equation in the form

$$\rho(\partial_t u + (u \cdot \nabla)u) = -\nabla p + f_F + f_L, \quad \nabla \cdot u = 0, \quad (30)$$

where $\rho$ means the mass density, which is constant, $p$ the hydrodynamic pressure, and $f_F$ and $f_L$ stand for the frictional force and the Lorentz force exerted on a unit
volume, respectively. From (30) we obtain the balance of the kinetic energy,
\[
\frac{d}{dt} \int_V \frac{\rho}{2} \mathbf{u}^2 \, dv = - \int_{\partial V} (p + \frac{\rho}{2} \mathbf{u}^2)(\mathbf{u} \cdot d\mathbf{s}) + \int_V \mathbf{u} \cdot \mathbf{f}_r \, dv + \int_V \mathbf{u} \cdot \mathbf{f}_L \, dv, 
\]  
(31)
where \( V \) means a volume that does not move, \( \partial V \) its surface and \( d\mathbf{s} \) a vectorial surface element pointing outward.

We identify now the volume \( V \) with the volume of a loop containing the central or helical channels outside the pump. Then \( \partial V \) corresponds to the walls of the loop and to two cross-sections of it at the two connections with the pump. There are three causes for the change of the kinetic energy in this volume, which are described by the three terms on the right-hand side of (31). The first cause is the pressure difference created by the pump, which is connected with the surface integral. Since \( \mathbf{u} \) is everywhere tangential to the walls only the mentioned cross-sections contribute to the integral. We assume that the two cross-sections are equal to each other. Then this integral describes just the work done by the pressure difference. The second and third causes are the frictional force and the Lorentz force the work of which is described by the two volume integrals.

Consider, for instance, the loop with the central channels, put \( \mathbf{u} = \mathbf{u}_C \) and use (4). Then it can be easily concluded that
\[
\frac{d}{dt} \int_V \frac{\rho}{2} \mathbf{u}^2 \, dv = \frac{\rho l_C}{\gamma_C s_C} V_C \frac{dV_C}{dt}, 
\]
where \( l_C \) is again the total length of the loop, \( s_C \) its cross-section, and \( \langle \cdots \rangle_C \) means the average over the volume of the loop. We adopt the reasonable assumption that the pressure \( p \) is constant across each of the two equal cross-sections at the connections with the pump and note that the integral \( \int_{\partial V}(\mathbf{u} \cdot d\mathbf{s}) \) over such a cross-section is, apart from the sign, equal to \( V_C \). We further utilize that the surface integral \( \int_{\partial V}(\rho/2)\mathbf{u}^2(\mathbf{u} \cdot d\mathbf{s}) \), which describes the flux of kinetic energy out of the loop, vanishes. Then we have
\[
- \int_{\partial V} (p + \frac{\rho}{2} \mathbf{u}^2)(\mathbf{u} \cdot d\mathbf{s}) = V_C P_C. 
\]
(33)
Finally we may write
\[
\int_V \mathbf{u} \cdot \mathbf{f}_r \, dv = V_C \langle \mathbf{u}_C \cdot \mathbf{f}_r \rangle_C l_C, 
\]
\[
\int_V \mathbf{u} \cdot \mathbf{f}_L \, dv = V_C \langle \mathbf{u}_C \cdot \mathbf{f}_L \rangle_C l_C, 
\]
(34)
where \( \langle \cdots \rangle_C \) means the average over that part of the loop which is penetrated by the magnetic field and \( l_C \) the length of this part. Starting from the balance (31), interpreting its terms in this way and putting
\[
\langle \mathbf{u}_C \cdot \mathbf{f}_r \rangle_C l_C = -R_C, \quad \langle \mathbf{u}_C \cdot \mathbf{f}_L \rangle_C l_C = -L_C
\]
(35)
we arrive immediately at the first relation of (2). Concerning a loop with helical channels we may proceed in the same way and arrive so at the second relation of (2).

For a rigid-body motion of the fluid \( \tilde{\mathbf{u}}_C \) and \( \tilde{\mathbf{u}}_H \) inside the spin-generators are, apart from signs, given by (8) with \( u(\phi)/u_0 = \omega(\phi)/\omega_0 = 1 \). If we ignore possible deviations from that in the tubes connecting the spin-generators inside the dynamo module and the latter with the pumps we find
\[
\gamma_C = 1, \quad \gamma_H = \frac{(\omega_1 + \omega_2)^2 + (h/\pi)^2}{2(\omega_1^2 + \omega_2^2) + (h/\pi)^2}.
\]
(36)
Taking the realistic values \( g_1 = 0.05 \text{ m}, \ g_2 = 0.105 \text{ m} \) and \( h = 0.19 \text{ m} \) we have \( \gamma_H = 0.9015 \).

**Appendix B. Calculation of the magnetic pressure drops \( L_C \) and \( L_H \)**. Let us consider the equations (12) for \( \Psi \) and \( B'_x \) under the assumption that \( u \) is non-zero only in \( 0 \leq x, y \leq a \) but equal to zero in all other space and use again the cylindrical coordinates \( \rho, \varphi \) and \( z \). Then \( \Psi \) and \( B'_x \) have to vanish as \( \rho \to \infty \). We put \( u_\varphi = 0, \ u_\varphi = -\omega(\rho)\varphi \) and \( u_z = -u(\rho) \). We may, of course, expand \( \Psi \) and \( B'_x \) in Fourier series with respect to \( \varphi \), that is, represent them as superpositions of components varying with \( \varphi \) like \( \exp(i m \varphi) \). Clearly the equations following from (12) for the individual Fourier components contain no couplings between components differing in \( m \). In addition, the equations for the components with \( |m| \neq 1 \) are homogeneous. The inhomogeneity in the case \( |m| = 1 \) results from the fact that \( \overline{B}_x \) and \( \overline{B}_\varphi \), which occur on the right-hand sides of (12), are connected with \( \overline{B}_x \) and \( \overline{B}_y \) via \( \cos \varphi \) and \( \sin \varphi \). As can be easily followed up all Fourier components with \( |m| \neq 1 \) have therefore to vanish. So we put without loss of generality

\[
\Psi = \frac{a}{2} R((\overline{B}_x - i \overline{B}_y) \psi(\rho) \exp(i \varphi)) , \quad B'_x = R((\overline{B}_x - i \overline{B}_y) b(\rho) \exp(i \varphi)) .
\]

(37)

We further introduce a dimensionless radius \( \xi \) and dimensionless quantities \( \hat{\omega} \) and \( \hat{u} \) formed after the pattern of magnetic Reynolds numbers by

\[
\xi = \frac{2 \rho}{a} , \quad \hat{\omega} = \frac{\omega a^2}{4 \eta} , \quad \hat{u} = \frac{u a}{2 \eta} .
\]

(38)

With equations (12) and (37) we obtain for \( \psi \) and \( b \), now considered as functions of \( \xi \),

\[
D \psi + i \hat{\omega} \psi = \hat{\omega} \xi \psi , \quad D b + i \hat{\omega} b = -\frac{d \hat{u}}{d \xi} \left( \frac{i}{\xi} \psi - 1 \right) ,
\]

(39)

where

\[
D f = \frac{d}{d \xi} \left( \frac{1}{\xi} \frac{d}{d \xi}(\xi f) \right) .
\]

(40)

In our first approximation, in which \( f \) is linear in \( u \), the terms with \( \hat{\omega} \) on the left-hand sides of the equations (39) as well as the term with \( \psi \) on the right-hand side of the second one have to be cancelled. Then the solution of (39) reads simply

\[
\psi(\xi) = -\frac{1}{2 \xi} \left( \int_{\xi_1}^{\xi} \hat{\omega}(\xi') \xi'^3 d\xi' + \xi^2 \int_{\xi_1}^{1} \hat{\omega}(\xi') \xi'^4 d\xi' \right) \quad \xi_1 = \frac{\hat{u}_1}{\hat{u}_C} , \quad \xi_2 = \frac{\hat{u}_H}{\hat{u}_C} ,
\]

(41)

Proceeding to the general case with respect to the magnitude of \( u \) but restricting ourselves to rigid-body motions of the fluid we put now \( \hat{\omega} = 0 \) and \( \hat{u} = \hat{u}_C \) in \( \xi < \xi_1 \), \( \hat{\omega} = \hat{\omega}_H \) and \( \hat{u} = \hat{u}_H \) in \( \xi_1 < \xi < \xi_2 \), and \( \hat{\omega} = \hat{u} = 0 \) in \( \xi > \xi_2 \), where \( \hat{u}_C, \hat{\omega}_H \) and \( \hat{u}_H \) are constants and \( \xi_1 \) and \( \xi_2 \) correspond to \( g_1 \) and \( g_2 \). It is easy to give then the general solutions of (39) in the three \( \xi \)-intervals. Considering that all components of the mean magnetic field and the tangential components of the mean electric field have to be continuous across the surfaces \( \xi = \xi_1 \) and \( \xi = \xi_2 \) it follows that \( \psi, d\psi/d\xi, b \) and \( \xi db/d\xi + \hat{u}(i \psi - \xi) \) have to be continuous, too.
Fitting the solutions for the three ξ-intervals in that sense we find

\[
\psi(\xi) = \begin{cases} 
(\dot{\psi}(\xi) - i\xi_1)\xi/\xi_1 & \text{for } 0 \leq \xi \leq \xi_1 \\
\dot{\psi}(\xi) - i\xi & \text{for } \xi_1 \leq \xi \leq \xi_2 \\
(\dot{\psi}(\xi_2) - i\xi_2)\xi_2/\xi & \text{for } \xi_2 \leq \xi 
\end{cases}
\]

and

\[
b(\xi) = \begin{cases} 
\dot{b}(\xi_1)\xi/\xi_1 & \text{for } 0 \leq \xi \leq \xi_1 \\
\dot{b}(\xi) & \text{for } \xi_1 \leq \xi \leq \xi_2 \\
\dot{b}(\xi_2)\xi_2/\xi & \text{for } \xi_2 \leq \xi 
\end{cases}
\]

(42)

\[
\tilde{\psi}(\xi) = \chi_2(\xi, \xi_1)
\]

\[
\tilde{b}(\xi) = -\frac{1}{2}(\hat{u}_H - \tilde{u}_C)\chi_2(\xi, \xi_1)\chi_0(\xi, \xi_2)/\xi_1 + \tilde{u}_H\chi_2(\xi, \xi_1)/\xi_2)
\]

\[
\chi_m(\xi, \xi') = -\frac{2i}{\kappa} \frac{J_1(\kappa\xi)N_m(\kappa\xi') - N_1(\kappa\xi)J_m(\kappa\xi')}{J_2(\kappa\xi_1)N_m(\kappa\xi_2) - N_2(\kappa\xi_1)J_m(\kappa\xi_2)}, \quad \kappa = \sqrt{\omega_h}.
\]

The \(J_m\) and \(N_m\) are Bessel functions of first and second kind, respectively, and \(\kappa\) is fixed so that \(\Re(\kappa) \leq 0\).

In order to determine the averaged quantities \(\langle \hat{u}_C \cdot \Gamma \rangle_C^m\) and \(\langle \hat{u}_H \cdot \Gamma \rangle_H^m\) it is useful to take first the averages over \(\varphi\), which concerns in fact only \(f_\varphi\) and \(f_z\). Then the remaining averaging over \(\xi\) can be carried out easily.

We denote averages over \(\varphi\) by \(\{ \cdots \}\). In the first approximation a straightforward calculation starting from (9), (11), (37) and (41) yields

\[
\{ f_\varphi \} = \frac{B^2}{\mu a} \hat{\omega} \xi, \quad \{ f_z \} = \frac{B^2}{\mu a} \hat{u}_0.
\]

(43)

Specifying finally \(\omega\) and \(u\) according to (1.23), which means \(\hat{u}_H = \hat{\omega}_h h/\pi a\), we find

\[
\langle \hat{u}_C \cdot \Gamma \rangle_C^m = -\frac{\sigma V_C}{2\gamma_C s_C} B^2 \psi_C, \quad \{ (\gamma_C^m)_C^m \} = (\hat{u}_C^C)_C^m,
\]

\[
\langle \hat{u}_H \cdot \Gamma \rangle_H^m = -\frac{\sigma V_H}{2\gamma_H s_H} B^2 \psi_H, \quad \{ (\gamma_H^m)_H^m \} = (\hat{u}_H^H)_H^m.
\]

(44)

In the general case with respect to the magnitude of \(u\) but for rigid-body motion of the fluid we have instead

\[
\{ f_\varphi \} = \frac{B^2}{\mu a} \hat{\omega} \xi \left( \psi_\varphi^\xi + (\psi_\varphi + \xi^2) \right), \quad \{ f_z \} = \frac{B^2}{\mu a} \frac{d}{d\xi} \left( (\psi_\varphi + \xi) b_r - \psi_r b_1 \right)
\]

(45)

and

\[
\langle \hat{u}_C \cdot \Gamma \rangle_C^m = -\frac{\sigma V_C}{2\gamma_C s_C} B^2 \psi_C, \quad \langle \hat{u}_H \cdot \Gamma \rangle_H^m = \frac{\sigma V_H}{2\gamma_H s_H} B^2 \psi_H
\]

with \(\psi_C\) and \(\psi_H\) given by

\[
\psi_C = \frac{2}{\hat{u}_C \xi_1} \left( (\psi_\varphi + \xi) b_r - \psi_r b_1 \right)_{\xi = \xi_1}
\]

\[
\psi_H = \frac{4}{(\xi_2 - \xi_1)(\xi_2^2 + \xi_2^2 + 2(h/\pi a)^2)} \left( \int_{\xi_1}^{\xi_2} \left( \psi_\varphi^\xi + (\psi_\varphi + \xi^2) \right) \xi d\xi \right)
\]

(47)

+ \frac{1}{\hat{u}_H} \left( \frac{h}{\pi a} \right)^2 \left( (\psi_\varphi + \xi) b_r - \psi_r b_1 \right)_{\xi = \xi_2}^{\xi = \xi_1}.

Of course, \(\gamma_C^m\) and \(\gamma_H^m\) have now to be specified to rigid-body motions. In or-
order to determine $\psi_C$ and $\psi_H$ as functions of $V_C$ and $V_H$ we have to put $\dot{u}_C = 2V_C/\eta \pi a \xi_l^2$, $\dot{u}_H = 2V_H/\eta \pi a (\xi_2 - \xi_1^2)$ and $\dot{u}_H = 2V_H/\eta \pi a (\xi_2 - \xi_1^2)$.

**Appendix C. Concerning the relations (5) and (6) and the numerical values of the coefficients given in (16).** In view of the pumps and of the hydraulic resistances we have used the technical report by Stieglitz and Müller [7], referred to as SM in the following. Further we have chosen for the radius of the central channel and the outer radius of the helical channel of a spin-generator $g_1 = 0.05 \text{ m}$ and $g_2 = 0.105 \text{ m}$, for their cross-sections $s_C = 7.85 \cdot 10^{-2} \text{m}^2$ and $s_H = 9.74 \cdot 10^{-3} \text{m}^2$, for the lengths of a central and a helical channel $l'_C = 0.71 \text{ m}$ and $l'_H = 1.95 \text{ m}$, for the mass density, the kinematic viscosity and the electrical conductivity of liquid sodium $\rho = 923 \text{kg/m}^3$, $\nu = 6.6 \cdot 10^{-7} \text{m}^2/\text{s}$ (both apply to a temperature of $120^\circ \text{C}$) and $\sigma = 8.0 \cdot 10^6 \text{S/m}$ (in agreement with $\eta = 0.1 \text{m}^2/\text{s}$, see Section I.5.1).

The relations (5) and the numerical values for $P^0_C$, $P^0_H$, $c_{PC}$ and $c_{PH}$ apply to the electromagnetic pumps used in the experiment and are taken from Fig. SM.3.5 (see also Section SM.B.1.1).

The first relation (6) follows from the relation (SM.3.7) (see also Appendix SM.B.1.2) with proper specification of the coefficients $\zeta$, for which relations given by Idelchik [1] were used. For the determination of the values of the coefficients $R^0_C$, $\rho_{RC}$ and $\epsilon_{RC}$ in addition to the parameters specified above a relative roughness of the surface of the channel of $10^{-4}$ was assumed.

The second relation (6) and the value of $R^0_H$ are direct consequences of (SM.3.6) (see also Appendix SM.B.1.1).

The coefficients $\kappa_C$ and $\kappa_H$ have been calculated on the basis of (3) with $\gamma_C = 1$ and $\gamma_H = 0.9015$. Considering the tubes between the dynamo module and the pumps we have chosen $l_C$ somewhat larger than $52 l'_C$ and $l_H$ somewhat larger than $26 l'_H$, more precisely $l_C = 40 \text{ m}$ and $l_H = 55 \text{ m}$. In this way the values of $\kappa_C$ and $\kappa_H$ given with (16) were obtained.

Likewise $c_{LC}$ and $c_{LH}$ were calculated on the basis of (15) with $\gamma^m_C = 1$ and $\gamma^m_H = 0.9015$. We have chosen $l^m_C = 52 l'_C$ and $l^m_H = 26 l'_H$, which leads to the values of $c_{LC}$ and $c_{LH}$ given with (16).

**Appendix D. Concerning the parameters $\lambda^P$ and $\lambda^T$.** The relation (21) with $\zeta^P \hat{\zeta}^T$ replaced by $(C^*)^{-2}$ represents the solutions of

$$\lambda^2 + (\lambda^P + \lambda^T)\lambda + \lambda^P \lambda^T (1 - (C/C^*)^2) = 0. \quad (48)$$

Writing down this equation twice, with $C = C_1$ and $\lambda(C_1) = \lambda_1$ and with $C = C_2$ and $\lambda(C_2) = \lambda_2$, we obtain two linear algebraic equations for $\lambda^P + \lambda^T$ and $\lambda^P \lambda^T$, which can easily be solved. In this way we find

$$\lambda^P, T = \frac{\lambda_1^2 \gamma_2 - \lambda_2^2 \gamma_1}{2(\lambda_2 \gamma_1 - \lambda_1 \gamma_2)} \left( 1 \pm \sqrt{1 - \frac{4\lambda_1 \lambda_2 (\gamma_2 - \gamma_1)(\gamma_2 \gamma_1 - \lambda_1 \gamma_2)}{(\lambda_1^2 \gamma_2 - \lambda_2^2 \gamma_1)^2}} \right) \quad (49)$$

Taking now from the curve of Fig. 1.22 showing $\lambda = \lambda(C)$ for $m = 1$ that $\lambda_1 = -2.06 \text{sec}^{-1}$ for $C_1 = 0$ and $\lambda_2 = 3.04 \text{sec}^{-1}$ for $C_2 = 12$, and considering that $C^* = 7.276$ we obtain $\lambda^P = 2.06 \text{sec}^{-1}$ and $\lambda^T = 30.81 \text{sec}^{-1}$. Specified in this way, relation (21) for $\lambda$ is in $0 \leq C \leq 25$ indeed in fair agreement with the curve mentioned. We note that the value of $\lambda^T$ depends critically on $\lambda_2$ and $C_2$ but, on the other hand, the approximation for $\lambda$ depends only weakly on the value of $\lambda^T$.

**Note added in proof.** The results for the functions $\psi_C$ and $\psi_H$ occurring in (14) which are represented in Figs. 1 and 2 have been determined on the basis of an
analytical solution of equations (10) for a single spin–generator, that is, ignoring the influence of the neighboring ones. In between they were also calculated from numerical solutions of (10) for an infinite array of spin–generators; see [6]. I turned out that, compared to the case of a single spin–generator, both functions decay less rapidly with \( V_H \), that is, the Lorentz force is to a lower degree reduced by the fluid flow. This is a consequence of the fact that in an array the magnetic flux expulsion from a spin–generator due to the rotational motion in the helical channel is less significant.

In the range of \( V_C \) and \( V_H \) which is of main interest for the experiment, say \( 100 \text{ m}^3/\text{h} < V_C, V_H < 120 \text{ m}^3/\text{h} \), the values of \( \psi_C \) for an array are by factors between 1.3 and 1.5 larger than those calculated for a single spin–generator, the values of \( \psi_H \) by factors between 1.4 and 1.6.

Replacing of the functions \( \psi_C \) and \( \psi_H \) for a single spin–generator by the more realistic ones for an array changes only a few of our above results, and these only slightly. The values of \( B_L, B^P \) or \( B^T \) that occur in Table 1 and in Figs. 3, 12 and 13 have to be reduced by factors between 0.8 and 0.9, and minor changes have to be done in Figs. 11 and 14. Apart from this the limits for \( k_C \) and \( k_H \) given in some of the figure captions have to be modified slightly.

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