Magnetic Field Decay in Neutron Stars. Analysis of General Relativistic Effects

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Abstract

An analysis of the role of general relativistic effects on the decay of neutron star's magnetic field is presented. At first, a generalized induction equation on an arbitrary static background geometry has been derived and, secondly, by a combination of analytical and numerical techniques, a comparison of the time scales for the decay of an initial dipole magnetic field in flat and curved spacetime is discussed. For the case of very simple neutron star models, rotation not accounted for and in the absence of cooling effects, we find that the inclusion of general relativistic effects result, on the average, in an enlargement of the decay time of the field in comparison to the flat spacetime case. Via numerical techniques we show that, the enlargement factor depends upon the dimensionless compactness ratio $\epsilon = \frac{2GM}{c^2R}$, and for ϵ in the range (0.3 , 0.5), corresponding to compactness ratio of realistic neutron star models, this factor is between 1.2 to 1.3. The present analysis shows that general relativistic effects on the magnetic field decay ought to be examined more carefully than hitherto. A brief discussion of our findings on the impact of neutron stars physics is also presented.

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I. INTRODUCTION

It is well known [1] that a magnetic field in a plasma of finite conductivity is subject to diffusion and dissipation. Diffusion leads to a spreading of inhomogeneities while dissipation is due to the Ohmic decay of the currents producing the field. More concretely, a magnetic field $\vec{B}(t,\vec{x})$ in a plasma of uniform conductivity σ evolves, in flat space-time, according to the following diffusion equation [1]:

$$\frac{\partial \vec{B}(t,\vec{x})}{\partial t} = \frac{c^2}{4\pi\sigma}\ \nabla^2 \vec{B}(t,\vec{x}). \tag{1.1}$$

Accordingly, if L is a typical length scale of the field structure, then it will decay-diffuse in a characteristic time scale $\tau_{\rm Ohm}$ given by: $\tau_{\rm Ohm} = \frac{4\pi\sigma L^2}{c^2}$. Depending upon the prevailing conditions, the Ohmic decay time $\tau_{\rm Ohm}$ can range from seconds, in the case of a copper sphere of radius of a few centimeters [1], up to $\tau_{\rm Ohm} = 10^{10}$ years or even much longer for astrophysical settings, as in the case of the sun [1] or a neutron star [2].

The interactions of large scale cosmic magnetic fields with plasmas is a problem of great importance in astrophysics and cosmology. A particularly thorny issue nowadays concerns the origin and maintenance of cosmic magnetic fields. Although large scale fields have been observed [3], a satisfactory explanation of their origin is still lacking. Peebles [4] considers the issue of the origin of the primordial magnetic field as one of the most important unsolved problems in cosmology. At the same time the gigantic field of the pulsars begs for an explanation [5]. The general consensus of the astrophysical community [6] is that such large scale fields have been generated via an episode of dynamo action [7], and then gradually suffer Ohmic decay due to the finite conductivity of the medium. It appears therefore that an understanding of the factors influencing the decay of large scale fields, combined with relevant observations, may offers important clues towards a better understanding of the initial scale involved as well as clues regarding its origin.

In neutron stars the decay of the magnetic field is an issue of out most importance by itself [8] and accordingly there has been an intense effort by astrophysicists to understand the factors governing this decay. As far as we are aware all theoretical modeling of magnetic field decay in neutron stars utilized the familiar flat space time form of Maxwell's equations (an exception to this rule constitutes the recent work of ref. [9]). Although the employment of such framework is a fruitful one and provides us with valuable informations, it altogether neglects the background curvature of the spacetime which for the case of neutron stars is not any longer weak. It would be worth to stress in that regard that curvature can modify considerably flat space-time solutions of Maxwell's equations. For instance, the reader may compare the solution describing a dipole magnetic field on a *Schwarzschild* background [10] to that of a flat space time. The presence of the logarithmic term in the former (see eqs. 3.17 further below) is a sole consequence of the non vanishing curvature. This example suggests that the role of the spacetime curvature on the decay process of magnetic fields ought to be examined more thoroughly than hitherto. In that respect, we are aware only of the recent work of Sengupta, [9], where an investigation of general relativistic effects in the magnetic field decay of neutron stars have been attempted. However this work is restricted to the study of magnetic fields confined only to the outermost layers of a neutron star and furthermore it is assumed that those outermost layers (and thus also the magnetic field $\vec{B}$), are embedded on a *Schwarzschild* background geometry. Thus strictly the framework of ref. [9], deals exclusively with magnetic decay on a *Schwarzschild* background. In addition to those approximations and according to the sentence following Eq. 15 of Sengupta's second work, the author fails to include general relativistic effects on the outer boundary condition for matching the inner field with the outer vacuum dipolar field across the surface of the star. In contrast, in the present work, a broad framework dealing with general relativistic effects on the magnetic field decay on an arbitrary static geometry and with proper allowance of the correct general relativistic inner and outer boundary conditions is presented. Moreover, and in contrast to the approach of ref. [9], we formulate the entire problem avoiding the introduction of a vector potential and the associated ambiguities. Our analysis shows that general relativistic effects [11] can influence the field decay, but the precise manner that this influence manifests itself depends upon the class of observers called in to describe the field decay. For the magnetic field of a non rotating neutron star it is natural to describe the field decay relative to the class observers that find themselves at rest relative to the star, ie the class of Killing observers. Relative to such observers, we find that relativistic effects are influencing the field decay via two major modes: the gravitational red shift as well as the intrinsic curved geometry of the spatial sections constituting the rest space of the Killing observers. Subsequent numerical analysis shows that the red shift factor is the dominant one in slowing down the field decay. Overall we find that the inclusion of relativistic effects make the decay time of the field larger than, but of the same order of magnitude, as in flat space-time. Nevertheless the preliminary study of the present paper utilizing a simple non rotating neutron star models suggests that general relativistic effects should be given further considerations. We explicitly illustrate the impact of relativistic effects upon the magnetic field decay, by examining the evolution of a magnetic field permeating a constant density neutron star, first in their presence and secondly without them. Although for both treatments we have obtained exponential decays, the decay time in the presence of relativistic effects, on the average, is enlarged by a factor that depends crucially upon the value of the compactness ratio $\epsilon = \frac{2GM}{c^2R}$. Specifically for values of ϵ in the domain (0.3 , 0.5), characterizing realistic neutron star models, we find that the decay time is 1.2 to 1.3 larger than the corresponding flat decay time, while for higher values of ϵ, it can be larger. We may add parenthetically that the term "average" increase in the decay time, is explained in details in section (IV) of the paper.

The present paper is organized as follows: In the following section, starting from Maxwell's equations on a static spacetime we first derive the relevant induction equation taking into account the curved nature of the background spacetime geometry. It should be stressed however that the employment of a static geometry does not leave room for incorporating gravitomagnetic (Lense-Thirring) effects in the induction equation, as the latter would manifest themselves relative to non static backgrounds, but we do hope to present such analysis in a future work. In section III, we specialize the induction equation to a simple neutron star model and a detailed analysis of the content of the induction equation is presented. In the same section the sensitive issue of the boundary conditions accompanying the induction equation is also addressed. In the section IV, we discuss numerical solutions of the curved spacetime induction equation and an assessment of the relativistic factors influencing the field decay is discussed. Furthermore in the same section, a comparison of the field decay in curved and flat spacetime is also presented. In the concluding section, a brief discussion of the physical implications of our results to neutron stars physics is presented and possible extension of the present work is outlined. Finally we have included an Appendix where a few intermediate calculations leading to the main equations of section II are presented.

II. INDUCTION EQUATION ON A STATIC BACKGROUND GEOMETRY

Maxwell's equations, in covariant form, are as follows [12,13]:

$$\nabla^{\alpha} F_{\alpha\beta} = -\frac{4\pi}{c} J_{\beta} \tag{2.1a}$$

$$\nabla_{[\alpha} F_{\beta\gamma]} = 0 \tag{2.1b}$$

where $F_{\alpha\beta} = -F_{\beta\alpha}$, J_{α} and ∇ are the coordinate components of the Maxwell tensor, the conserved four current and the derivative operator respectively. Given a solution $F_{\alpha\beta}$ of the above eqs., an observer with four velocity U^{α}, $U^{\alpha}U_{\alpha} = -1$, measures electric and magnetic fields (E, B) with corresponding coordinate components given respectively by:

$$E_{\alpha} = F_{\alpha\beta} U^{\beta} \quad , \quad B_{\alpha} = -\frac{1}{2} \epsilon_{\alpha\beta}{}^{\gamma\delta} F_{\gamma\delta} U^{\beta} \tag{2.2}$$

where $\epsilon_{\alpha\beta\gamma\delta}$ stands for the four-dimensional Levi-Civita tensor density [14]. We shall be concerned in the present paper with particular solutions of 2.1 where the current J is described by the following relativistic extension of Ohm's law, as it was first formulated by Weyl [15]:

$$J^{\alpha} = \sigma g^{\alpha\beta} F_{\beta\gamma} V^{\gamma} \tag{2.3}$$

where in above equation, (V^{α}, σ) stand for the four velocity of a conducting neutral plasma and its scalar electrical conductivity [16] respectively. Although eqs. 2.1 to 2.3 are valid for any kind of background geometries and plasmas characterized by arbitrary four velocity and conductivity, hereafter we shall restrict our consideration to background geometries that are globally static. Staticity in turn allows us to select coordinates so that the spacetime geometry can be written in the form (see for instance discussion in ref. [12,13]):

$$ds^2 = -e^{2\Phi}(dx^o)^2 + \gamma_{ij}dx^i dx^j \tag{2.4}$$

where $x^o = ct$, γ_{ij} are functions of the the spatial coordinates x^i, $(i = 1, 2, 3)$, and by ξ denote the hypersurface orthogonal timelike Killing vector field obeying: $\xi_\alpha \xi^\alpha = -e^{2\Phi}$. For the above form of the line element, Maxwell's equations 2.1 and the current conservation law $\nabla_\alpha J^\alpha = 0$ can be re-written in an equivalent form involving only the components (E^i, B^i) of the electric and magnetic fields respectively, as well as the charge density $c\rho = -U_\mu J^\mu$ and spatial current density J^i as measured by the Killing observers [17,18]. More precisely if by U^μ we denote their four velocity then eqs. 2.1 yield the following equivalent set (see Appendix for details, or ref. [17,18]):

$$D_i E^i = 4\pi\rho \qquad , \qquad D_i B^i = 0 \tag{2.5a}$$

$$\epsilon^{ijk} D_j (Z B_k) = \frac{4\pi}{c} Z J^i + \frac{\partial E^i}{\partial x^o} \tag{2.5b}$$

$$\epsilon^{ijk} D_j (Z E_k) = -\frac{\partial B^i}{\partial x^o} \tag{2.5c}$$

$$U^\mu \frac{\partial (c\rho)}{\partial x^\mu} + D_i J^i + J^i D_i log Z = 0 \tag{2.5d}$$

where in above D stands for the covariant derivative operator associated with γ, ϵ^{ijk} represent the (coordinate) components of the three dimensional totally antisymmetric Levi-Civita tensor density defined on the $x^o = const$ slices and $Z = (-\xi^\alpha \xi_\alpha)^{\frac{1}{2}} = e^\Phi$ is the red shift factor which in the language of in the $3+1$ approach to spacetime or (and) electrodynamics, is also refered as the lapse function [18].

With Maxwell's eqs. in the above form we can derive an induction equation by repeating the same steps leading to the derivation of its flat counterpart (see for example discussion in [1]). For a plasma at rest relative to the Killing observers, combined with Ohm's law and the MHD approximation (i.e. neglecting the displacement current [19] from the right hand side of 2.5b), one obtains from 2.5-abc the following form of the generalized induction equation:

$$\frac{\partial B^i}{\partial x^o} + \epsilon^{ijk} D_j \left[\frac{c}{4\pi\sigma} \epsilon_k{}^{lm} D_l (Z B_m) \right] = 0 \tag{2.6}$$

This last equation describes the time evolution of a magnetic field configuration that finds itself in a conducting medium. In principle one could write down the explicit form of the dynamical evolution equation once a choice of background geometry has been made. However before we do so, we would like to make a further specialization of the eqs. 2.5 and 2.6, so that their interrelationship to the familiar flat space three plus one formalism of Maxwell's eqs. is more transparent. Here, following the spirit of [18] and particularly [20], we shall sacrifice the manifest three-covariance of eqs. 2.5 and 2.6 with respect to arbitrary coordinate transformations of the $t = const$ sections, for the benefits of practical usefulness. As was pointed out in ref. [18,20], if one defines suitably the components of (E, B) and under some weak constraints upon the spacetime geometry, then Maxwell's equations can be recast in a more "user friendly" form. This new form employs concepts familiar from the language of the three dimensional vector analysis expressed in orthogonal curvilinear coordinates and such approach to curved spacetime electrodynamics is particularly useful for astrophysical purposes. Having in mind further astrophysical applications of our results we shall recast eqs. 2.5-abc in such a form. Such form requires that the geometry of the spacetime permits the introduction of coordinates so that the spatial three element $ds^2_{(3)}$ of 2.4 could be recast in the following form:

$$ds^2_{(3)} = h_1^2 (dx^1)^2 + h_2^2 (dx^2)^2 + h_3^2 (dx^3)^2 \tag{2.7}$$

where the scale factors $h_i = h_i(x^1, x^2, x^3)$ are for the moment arbitrary functions of (x^1, x^2, x^3). In the Appendix, (see also [18,20]) we show that for such geometries eqs. 2.5 can be written in the following form:

$$\nabla\cdot\vec{E} = 4\pi\rho \ , \quad \nabla\cdot\vec{B} = 0 \tag{2.8a}$$

$$\vec{\nabla}\times(Z\vec{B}) = \frac{4\pi}{c}Z\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \tag{2.8b}$$

$$\vec{\nabla}\times(Z\vec{E}) = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \tag{2.8c}$$

$$\nabla\cdot\vec{J} + \vec{J}\cdot\nabla(logZ) = 0 \tag{2.8d}$$

where we have written the current conservation law for an electrically neutral plasma and in above equations the symbols ($\nabla\cdot$, $\vec{\nabla}\times$, ∇) stand for the divergence, curl and gradient operators respectively, expressed entirely in terms of the scale factors h_i (see Appendix for their explicit representation). We also remind the reader that all vector components in equations 2.8 are physical frame components taken with respect to the field of orthonormal frames $e_i = \frac{1}{h_i}\frac{\partial}{\partial x^i}$, $(i = 1, 2, 3)$, naturally singled out by the line element 2.7. Using now eqs. 2.8, or directly from eq. 2.6, upon eliminating the coordinate components of $\vec{B}$ in favor of its frame components, the induction equation 2.6 takes the following form:

$$\frac{1}{c}\frac{\partial\vec{B}}{\partial t} + \vec{\nabla}\times\left[\frac{c}{4\pi\sigma}\vec{\nabla}\times(Z\vec{B})\right] = 0 \tag{2.9}$$

Equations 2.8 and 2.9 are the main equations of this section. In the special case of a *Schwarzschild* background, naturally they are reduced to those of ref. [20], and in the case of a plasma of uniform conductivity the generalized induction eq. 2.9 reduces to eq. 1.1 in the limit of flat space.

III. MAGNETIC FIELD DECAY INTERIOR TO NEUTRON STARS

In the neutron star's interiors the MHD approximation is well justified [19] and we shall explore the content of the relativistic induction equation 2.9, by applying it to study the evolution of magnetic fields associated with neutron stars. Since the main purpose of the present work is to investigate the impact of the spacetime curvature upon the magnetic field decay, as a first preliminary step we shall adopt a rather simplified neutron star model. The chosen model primarily avoids technicalities that may obscure the issue at hand but at the same time it shows clearly the potential impact of the curvature on the magnetic field decay. Accordingly, and to avoid laborious numerical computations, we shall ignore the rotation of the neutron star and thus shall adopt as the background geometry a non-singular, static and spherically symmetric one. Hence, the scale factors of eq. 2.7 will be taken as:

$$h_r^2 = \left(1 - \frac{2Gm(r)}{rc^2}\right)^{-1} = \left(1 - \frac{2M(r)}{r}\right)^{-1} \quad , \quad h_\theta^2 = r^2 \quad , \quad h_\phi^2 = r^2 \sin^2\theta \tag{3.1}$$

while for the moment the lapse or red shift factor $Z = Z(r) = e^{\Phi(r)}$ and the "mass function" $m = m(r)$ are arbitrary functions of the radial coordinate.

We shall begin our analysis of the magnetic field decay by assuming that at some initial time t_o an axially symmetric distribution of a magnetic field $\vec{B}(t_o, r, \theta)$ permeates the entire star. We are not concerned here upon the mechanism that brought such a field into existence but rather we are interested in its evolution. Its evolution is considerably affected by the electrical conductivity σ, but as a part of the adopted simplified picture and in order to emphasize the effects of space-time curvature we shall take σ to be spherically symmetric and shall ignore any cooling effects that may influence its temporal evolution. For an axially symmetric field $\vec{B}$, it is convenient to decompose it into the so called poloidal $\vec{B}_{(p)}$ and toroidal part $\vec{B}_{(t)}$. In terms of the orthonormal basis vectors $(e_r,\ e_\theta,\ e_\phi)$ those parts are defined respectively by: $\vec{B}_{(p)} = B^r\vec{e}_r + B^\theta\vec{e}_\theta$ and $\vec{B}_{(t)} = B^\phi\vec{e}_\phi$ with (B^r, B^θ, B^ϕ) arbitrary functions of (t, r, θ) respectively. One can then easily conclude from the induction eq. 2.9 that, as long as the scalar conductivity is spherically symmetric, the toroidal and poloidal parts of $\vec{B}$, evolve independently of each other [21]. Such decoupling is rather convenient since it implies that if the initial distribution of the magnetic field is purely poloidal then it will not develop a toroidal component in the course of its evolution and vice versa. For simplicity, in the present paper we shall examine the effects of the spacetime curvature only on the evolution of a purely poloidal field $\vec{B}_{(p)} = B^r\vec{e}_r + B^\theta\vec{e}_\theta$. For such field $\vec{B}$, it follows from 2.8b that the current $\vec{J}$ is along the $\vec{e}_\phi$ direction, and thus the current conservation eq. 2.8d is identically satisfied. Furthermore via Ohm's law, and use of 2.8b (with the displacement current ignored), it follows that the electric field $\vec{E}$ is a purely toroidal and axisymmetric field and Gauss law $\nabla\cdot\vec{E} = 0$ is satisfied as well. Consequently, from the system of eqs. 2.8, we are left to satisfy the constraint $\nabla\cdot\vec{B} = 0$, solutions of which will be evolved by the induction eq. 2.9.

Taking into account the poloidal and axisymmetric nature of $\vec{B}$ as well as the formula of the div operator $\nabla\cdot$, listed in the Appendix, in view of the scale factors of 3.1, one easily finds that $\nabla\cdot B = 0$ implies:

$$\left(1 - \frac{2M}{r}\right)^{\frac{1}{2}} \frac{1}{r}\frac{\partial(r^2 B^r)}{\partial r} + \frac{1}{\sin\theta}\frac{\partial(B^\theta \sin\theta)}{\partial\theta} = 0 \tag{3.2}$$

We shall look for separable solutions of the above equations in the form:

$$B^r = F(t,r)\Theta_1(\theta), \quad B^\theta = G(t,r)\Theta_2(\theta) \tag{3.3}$$

with the functions F, G, Θ_1, Θ_2 to be determined. Substituting the above representations of $(B^r,\ B^\theta)$ in 3.2 and separating variables one gets the following equivalent system:

$$\left(1 - \frac{2M}{r}\right)^{\frac{1}{2}} \frac{1}{r}\frac{\partial(r^2 F)}{\partial r} - \lambda G = 0 \tag{3.4a}$$

$$\frac{1}{\sin\theta}\frac{\partial(\sin\theta\ \Theta_2)}{\partial\theta} + \lambda\Theta_1 = 0 \tag{3.4b}$$

where λ stands for a separation constant. The second equation can be solved in terms of the Legendre polynomials by taking $\lambda = l(l+1)$, $l = 0, 1, 2...$, and:

$$\Theta_2 = \sin\theta\frac{dP_l(y)}{dy}, \quad \Theta_1 = -P_l(y), \quad y = \cos\theta \tag{3.5a}$$

On the other hand, for such λ, eq. 3.4a is satisfied provided, for $l\neq 0$, one chooses $G(r,t)$ in the following form;

$$G(t,r) = \frac{1}{l(l+1)}\left(1 - \frac{2M}{r}\right)^{\frac{1}{2}} \frac{1}{r}\frac{\partial(r^2 F)}{\partial r} \tag{3.5b}$$

We shall disregard the $l = 0$ mode since, as it is clear form above, it corresponds to a monopole field B. With the exclusion of monopole fields, the components of an arbitrary axisymmetric poloidal field can be written as a superposition of "l-poles" in the form:

$$B^r(t,r,\theta) = -\sum_{l=1}^{\infty} F_l(t,r)\, P_l(y) \tag{3.6a}$$

$$B^\theta(t,r,\theta) = \sum_{l=1}^{\infty} \frac{1}{l(l+1)} \left(1 - \frac{2M}{r}\right)^{\frac{1}{2}} \frac{1}{r}\frac{\partial(r^2 F)}{\partial r} \sin\theta\, \frac{dP_l(y)}{dy} \tag{3.6b}$$

To simplify algebra, and on physical grounds, we shall restrict our considerations to the detailed analysis of only the $l = 1$ mode. Such mode corresponds to a dipole field and such configuration is expected to be present and dominant within neutron stars. For $l = 1$, eqs. 3.6 yield:

$$B^r(t,r,\theta) = -F(t,r)\cos\theta, \quad B^\theta(t,r,\theta) = \frac{1}{2r}\left(1 - \frac{2M}{r}\right)^{\frac{1}{2}} \frac{\partial(r^2 F)}{\partial r} \sin\theta \tag{3.7}$$

where for notational simplicity we write here after F instead of F_1. On the other hand, for any poloidal axisymmetric field with components (B^r, B^θ), the induction equation 2.9 on the background geometry of 2.7, yields the following two non-trivial evolution equations:

$$\frac{\partial B^r}{\partial x^o} + \frac{1}{h_\theta h_\phi}\frac{\partial}{\partial\theta}\left[\frac{cA}{4\pi\sigma}\right] = 0 \tag{3.8a}$$

$$\frac{\partial B^\theta}{\partial x^o} - \frac{1}{h_r h_\phi}\frac{\partial}{\partial r}\left[\frac{cA}{4\pi\sigma}\right] = 0 \tag{3.8b}$$

where:

$$A = \frac{h_\phi}{h_r h_\theta}\left[\frac{\partial}{\partial r}\left(h_\theta B^\theta Z\right) - \frac{\partial}{\partial\theta}\left(h_r B^r Z\right)\right] \tag{3.8c}$$

When one now inserts in eq. 3.8a the explicit forms of the components of (B^r, B^θ) corresponding to a dipole field in the form shown in eq. 3.7, as well as the scale factors of eq. 3.1, then gets:

$$\frac{4\pi\sigma}{c}\frac{\partial F}{\partial x^o} = \left(1 - \frac{2M}{r}\right)^{\frac{1}{2}} \frac{1}{r^2}\frac{\partial}{\partial r}\left[Z\left(1 - \frac{2M}{r}\right)^{\frac{1}{2}} \frac{\partial(r^2 F)}{\partial r}\right] - \frac{2ZF}{r^2} \tag{3.9}$$

In arriving at the above equation we have taken explicitly into account the spherically symmetric nature of the scalar conductivity σ. We may point out that for non spherical σ, the right hand side of 3.9 contains gradients of σ along the meridian directions but for our simple neutron star model a spherical conductivity is rather adequate. On the other hand identical manipulations of (3.8b) leads to:

$$\frac{\partial}{\partial r}\left[r^2\frac{\partial F}{\partial x^o}-\frac{c}{4\pi\sigma}\left(1-\frac{2M}{r}\right)^{\frac{1}{2}}\frac{\partial}{\partial r}\left[Z\left(1-\frac{2M}{r}\right)^{\frac{1}{2}}\frac{\partial(r^2F)}{\partial r}\right]+2\frac{c}{4\pi\sigma}ZF\right]=0 \qquad (3.10a)$$

from which we infer that

$$r^2\frac{\partial F}{\partial x^o}-\frac{c}{4\pi\sigma}\left(1-\frac{2M}{r}\right)^{\frac{1}{2}}\frac{\partial}{\partial r}\left[Z\left(1-\frac{2M}{r}\right)^{\frac{1}{2}}\frac{\partial(r^2F)}{\partial r}\right]+2\frac{c}{4\pi\sigma}ZF=g(\theta,\phi,t) \qquad (3.10b)$$

where $g(\theta,\phi,t)$ an integration "constant". A comparison then between (3.9) and (3.10b) shows that necessary $g=0$. If we further define: $\hat{F}=r^2F$, then one finds either from 3.9 or 3.10b that $\hat{F}$ satisfies:

$$\frac{4\pi\sigma}{c}\frac{\partial\hat{F}}{\partial x^o}=\left(1-\frac{2M}{r}\right)^{\frac{1}{2}}\frac{\partial}{\partial r}\left[Z\left(1-\frac{2M}{r}\right)^{\frac{1}{2}}\frac{\partial\hat{F}}{\partial r}\right]-2\frac{Z}{r^2}\hat{F} \qquad (3.11)$$

The above equation essentially describes the evolution of the the dipole field components [22] as they are measured by the Killing observers. Linearity of Maxwell's and induction equation implies that 3.11 specifies a unique solution up to an arbitrary rescaling. This rescaling freedom will be fixed later on by a suitable matching of the interior dipole field to a corresponding asymptotically vanishing exterior dipole one.

Taking $Z=1, \quad M=0$ in eq. 3.11 one recovers the standard equation describing the evolution of a dipole axisymmetric poloidal field in flat space [23] namely:

$$\frac{4\pi\sigma}{c^2}\frac{\partial S}{\partial t}=\frac{\partial^2 S}{\partial r^2}-\frac{2S}{r^2} \qquad (3.12)$$

where in order to avoid confusion we have denoted the analogue of $\hat{F}(t,r)$ for the flat space time case by $S(t,r)$. The latter function often in the astrophysics literature is referred to as the Stoke's function [24]. In order to get some insights into the significance of the various terms appearing in (3.11), the general relativistic counterpart of 3.12, we shall rewrite the former in an equivalent form so that a clean comparison between the two could be afforded. Eliminating the (areal) radial coordinate r in favor of the physical proper radius $l(r)$ of the $r=constant$ spheres, via $dl=dr(1-\frac{2M}{r})^{-\frac{1}{2}}$, Eq. 3.11 takes then the following form:

$$\frac{4\pi\sigma}{c^2}\frac{\partial\hat{F}}{\partial t} = \frac{\partial}{\partial l}(Z\frac{\partial\hat{F}}{\partial l}) - \frac{2Z}{r(l)^2}\hat{F} \tag{3.13a}$$

A comparison then to eq. 3.12 shows that relative to the Killing observers, general relativistic effects can influence B-decay in three ways. Namely via the presence of the red shift factor Z, its gradient and as well as via the intrinsically curved nature of the rest space of the Killing observers, ie the $t = const$ hyperfaces. The latter manifest itself in 3.13a via the term $r(l)$, a term which in general satisfies $r(l) \neq l$, implying that that the rest spaces of the Killing observers are intrinsically curved. From the above mentioned three factors the spatial gradient of Z makes negligible contribution to the field decay and this has been verified numerically. Neglecting this gradient then equation 3.13a takes the following form:

$$\frac{4\pi\sigma}{c^2 Z(l)}\frac{\partial\hat{F}}{\partial t} = \frac{\partial^2\hat{F}}{\partial l^2} - \frac{2\hat{F}}{r(l)^2} \tag{3.13b}$$

In this form a clean comparison to equation (3.12) can be afforded. The right hand sides of the two equations involve physical spatial gradients and the differ only by terms of the order $O(\frac{2Gm}{c^2R})$. On the other hand their left hand sides as they stand cannot be compared. If however one reasonably replaces $Z(l)$ by some averaged value $< Z >$, then the left hand side of (3.13) involves also physical temporal gradients. In that event one gets a first flavor of the magnitude of the general relativistic effects. They modify the corresponding flat spacetime results by terms of order unity. Of course such conclusions has to be also documented at the solution level as well, and as we shall further ahead this indeed is the case.

Having thus identified the manner by which relativistic gravity effects the magnetic field decay, our assignment is now to access the relative importance of each of the above two factors. In the following section we shall do so by resorting to numerical computations. However, before we pass to that issue let us first record the suitable boundary conditions to be imposed upon the corresponding $S(t, r)$, $\hat{F}(t, r)$ in order to describe sensible physics. The required conditions for both, ie the flat and the general relativistic case, are drawn by demanding that the interior $\vec{B}$ ought to be a non singular field at all times and at all spatial points, and in addition it ought to join smoothly across the surface of the star to an exterior asymptotically vanishing dipole field. For the flat space case, taking $M = 0$ and $F(t,r) = -\frac{2S}{r^2}$ in eq. 3.7 one gets: $\vec{B} = \frac{2S}{r^2}\cos\theta\ \vec{e}_r - \frac{1}{r}\frac{\partial S}{\partial r}\sin\theta\ \vec{e}_\theta$ from which we infer that the magnitude $\vec{B}^2$ of the interior magnetic field is given by: $|\vec{B}|^2 = \frac{4S^2}{r^4}\cos^2\theta + \frac{1}{r^2}(\frac{\partial S}{\partial r})^2 \sin^2\theta$. Accordingly a regular field at the star's center requires $lim\frac{S(t,r)}{r^2}$ to be finite as the center of the star is approached. On the other hand an asymptotically vanishing dipole magnetic field in flat space due to a magnetic moment μ, is described by [1]: $\vec{B} = \frac{2\mu\cos\theta}{r^3}\ \vec{e}_r + \frac{\mu\sin\theta}{r^3}\ \vec{e}_\theta$. It follows then from the above expressions that a C^0 matching of the interior magnetic field to slow varying exterior dipole field [25], requires that across the star's surface, ie at radius R, $S(t,r)$ should satisfy:

$$R\,\frac{\partial S}{\partial r}\bigg|_R = -S \tag{3.14}$$

The relatively simple nature of (3.12) as well the simple form of the boundary-regularity conditions outlined above, permit us to construct exact closed form solutions. In fact, it is not difficult to verify that for the case of a star of a uniform conductivity σ, a sequence of exact solutions of eq. (3.12) obeying the above described conditions is given by [26]:

$$S_n(t,x) = \left[\frac{\sin(n\pi x)}{n^2\pi^2 x} - \frac{\cos(n\pi x)}{n\pi}\right]\ e^{-\frac{t}{\tau_n}} = \psi_n(x)\ e^{-\frac{t}{\tau_n}} \tag{3.15}$$

where $\tau_n = \frac{4\sigma R^2}{\pi c^2 n^2} = \frac{1}{n^2}\,\frac{\tau_{\rm Ohm}}{\pi^2}$, $x = \frac{r}{R}$ and n takes the values $(1, 2, 3....)$.

The above sequence of exact solutions offers a clear picture regarding the behavior of a magnetic field in a conducting medium that finds itself in a flat space time. Constructing for instance the field $\vec{B}_1(t,r,\theta)$ corresponding to $S_1(t,r)$, one immediately sees that an observer at fixed (r,θ) finds that the magnitude of $\vec{B}_1(t,r,\theta)$ decays exponentially with a characteristic e-folding time given by $\tau_1 = \frac{4\sigma R^2}{\pi c^2}$. On the other hand expanding an arbitrary initial field configuration $\vec{B}(t_0,r,\theta)$ in terms of the eigenfunctions $(\psi_n, n = 1,2..)$, one can easily see the spatial diffusion of the initial distribution. For a plasma characterized by an arbitrary σ, although the decay and diffusive nature of the initial $\vec{B}$ field remain intact, it is rather difficult to estimate analytically the characteristic decay time as well as to find out whether the decay will be channeled into an exponential phase. It is sufficient, however, to stress that as long as we are in flat spacetime the decay process is controlled by the conducting properties of the background medium and, of course, the length scale of the initial field distribution.

Let us now turn the discussion to the formulation of the appropriate conditions to be imposed on solutions of eq. 3.11. Since as already indicated in the introduction section, a dipole field on a *Schwarzschild* is modified considerably from its flat form and, as eq. 3.11 shows, relativistic effects modify the local behavior of the relativistic Stoke's function $\hat{F}(t,r)$, one expects modification of the boundary conditions as well. As far as the behavior of $\hat{F}(t,r)$ at the star's center is concerned, by arguing in the same manner as in the flat space case, a non singular dipole field requires $\hat{F}(t,r)$ to satisfy identical conditions at the star's center as its flat counterpart, namely $lim \frac{\hat{F}(t,r)}{r^2}$ should be finite as the center of the star is approached (after all, the principle of equivalence holds). Although this is the case at the star's center the corresponding boundary conditions at the star's surface are markedly different. Recalling that the frame component of the vector potential $A = A_\mu e^\mu = A_\phi e^\phi$ describing a magnetic dipole on a *Schwarzschild* background is described by [10]:

$$A_\phi = \frac{3\mu \sin\theta}{4M^2} \left[x \ln\left(1 - \frac{1}{x}\right) + \frac{1}{2x} + 1 \right] \tag{3.16}$$

where $x = \frac{r}{2M(R)}$, $R \leq r < \infty$, and μ is the dipole moment. From 3.16, one then obtains the corresponding field $\vec{B} = B^r\, \vec{e}_r + B^\theta\, \vec{e}_\theta$ where the physical components $(B^r,\, B^\theta)$ as measured by the Killing observers are given by:

$$B^r = \frac{2\mu\cos\theta}{r^3} \left[3x^3\, \ln(1 - x^{-1}) + 3x^2 + \frac{3}{2}x \right] \tag{3.17a}$$

$$B^\theta = -\frac{\mu\sin\theta}{r^3} \left[6x^3 (1 - x^{-1})^{\frac{1}{2}}\, \ln(1 - x^{-1}) + 6x^2 \frac{1 - \frac{1}{2x}}{(1 - x^{-1})^{\frac{1}{2}}} \right] \tag{3.17b}$$

A comparison of (3.17, ab)) with the corresponding eq. 3.7 and a C^0 matching between the two along the star's surface, requires that the exterior dipole magnetic moment μ should be identified with the generally slow time varying part of the function $\hat{F}(t,r)$ [25]. Moreover the gradient $\hat{F}(t,r)$ along the radial direction should obey:

$$R \left. \frac{\partial \hat{F}(t,r)}{\partial r} \right|_R = G(y)\hat{F}(t,R) \tag{3.18a}$$

with:

$$G(y) = y \, \frac{2y \ln(1-y^{-1}) + \frac{2y-1}{y-1}}{y^2 \ln(1-y^{-1}) + y + \frac{1}{2}} \tag{3.18b}$$

and $y = \frac{R}{2M(R)}$.

Thus the behavior of the interior dipole field in the presence of curvature is described by $\hat{F}(t,r)$ satisfying the differential eq. 3.11, subject to boundedness of $\frac{\hat{F}(t,r)}{r^2}$ as the star's center is approached, and additionally obeying 3.18a at its surface. Before we turn our discussion to the construction of solutions of eq. 3.11 subject to the above discussed conditions, we would like to write an explicit formula for the time evolution of $\hat{F}(t,r)$ under the assumption that geometry of the spacetime corresponds to a static spherically symmetric star, solution of Einstein's equations. We may recall that in the derivation of eqs. (3.11) and (3.18,ab) we assumed an arbitrary non-singular, static, spherically symmetric background geometry with the only constraint that it joins smoothly to an exterior *Schwarzschild* field across the surface of the star. Nowhere in the derivation we needed the explicit form of the $M(r) = \frac{Gm(r)}{c^2}$ nor the form of $Z = Z(r)$. Hereafter we shall become more explicit and shall take the background interior geometry to be a non singular solution of the coupled Einstein-perfect fluid system. As is well known, and under the assumption that the background Maxwell field makes negligible contribution to the structure of the star [27], static spherically symmetric, perfect fluid solutions of Einstein's equations imply satisfaction of the following differential equations between the metric functions $\Phi(r)$, $M(r)$, the hydrostatic pressure $P(r)$, and mass density $\rho(r)$ (see for instance [12,13]):

$$\frac{d\Phi}{dr} = \frac{M(r) + 4\pi r^3 P(r)}{r^2\left(1 - \frac{2M(r)}{r}\right)} \tag{3.19a}$$

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho \tag{3.19b}$$

$$\frac{dP(r)}{dr} = -(\rho + P)\frac{M(r) + 4\pi r^3 P}{r^2\left(1 - \frac{2M(r)}{r}\right)} \tag{3.19c}$$

Making use of those equations, and restoring the fundamental units, we obtain from 3.11 the following equation to be satisfied by the relativistic Stokes function:

$$\frac{4\pi\sigma}{c^2}e^{-\Phi(r)}\frac{\partial F}{\partial t} = \left(1 - \frac{2Gm(r)}{c^2 r}\right)\frac{\partial^2 F}{\partial r^2} + \frac{1}{r^2}\left[\frac{2Gm(r)}{c^2} + \frac{4\pi G}{c^2}r^3\left(\frac{P(r)}{c^2} - \rho(r)\right)\right]\frac{\partial F}{\partial r} - \frac{2}{r^2}F \tag{3.20}$$

where for typographical convenience we shall write here after $F(r,t)$ instead of $\hat{F}(r,t)$. The above equation via 3.7, describes the evolution of any axisymmetric, dipole, poloidal field $\vec{B}$ that finds itself interior to a spherical perfect fluid star. In the above form it includes all three relativistic factors influencing the field decay. Since the distributions of $(m(r),\ \ P(r),\ \ \rho(r))$ are related via Einstein's equations directly to the spacetime curvature, eq. 3.20 shows implicitly that the influence of spacetime curvature on the decay of the magnetic field is a real effect and cannot be removed via coordinate transformations. In principle one could insert in 3.20 the appropriate distributions of $m(r),\ \ P(r),\ \ \rho(r)$ resulting from integrating the Oppenheimer-Tolman-Volkov equation, specify $\sigma = \sigma(r,t)$, and construct the history of the $\vec{B}$-decay. We shall report elsewhere our findings of this rather laborious numerical integration [28]. For the purpose of the present paper we shall integrate 3.20 for a rather simple system introduced and discussed in the following section. The goal of this section is to show that the general relativistic eq. 3.20 (or its approximate forms corresponding to eq. (3.13b) under the assumption of a uniform conductivity admits decay modes analogous to those of the flat space case with one important difference: The corresponding e-fold decaying times are longer in the relativistic case. We interpret this amplification of the e-fold decay times as resulting from the non vanishing space time curvature. Unfortunately however it is not easy to construct analytically the exact decaying modes of the full relativistic system 3.20 or its approximate versions (3.13a), and thus we shall resort to numerical computations. The emphasis in those computations is the probing of the dependence of the corresponding e-folding times upon the value of the red shift factor or (and) upon the strength of the curvature of the spatial sections.

IV. MAGNETIC FIELD DECAY IN A CONSTANT DENSITY STAR. EXPLICIT RESULTS.

We shall consider in this section the decay of a magnetic field in a neutron star of constant density. The assumption of a constant density star, although not a very reliable approximation of a real neutron star, offers the advantage that the Einsteins equations can be solved analytically (see for instance [12,13]), and thus provides us with closed form expressions for the coefficients of the induction equation, eq. 3.20, and the boundary condition, eq. 3.18. In particular, in Box 23.2 of the ref. [13] the distribution of the various hydrodynamical and geometrical variables are plotted as functions of the areal coordinate r. As we have already discussed, the exact decaying modes of the flat space-time induction equation for a uniform conductivity are explicitly known (given by eq. 3.15), while the corresponding decaying modes of the full curved space-time equation 3.20 are presently unknown. We shall therefore resort to numerical techniques in an attempt to get insights into behavior of the space of solutions of eq. 3.20.

Viewed as an initial-boundary value problem, (3.20) is a diffusive initial value problem for which the standard numerical technique is the Crank-Nicholson implicit integration scheme (see for example ref.[29] for a description). We checked our numerical code by evolving the fundamental mode of the flat space-time case (ie take n=1 in eq. 3.15) and compare the numerical solution with the analytical one: we obtained an accuracy better than 1% until times up to $10 \cdot \tau_1$, with $\tau_1 = \frac{4\sigma R^2}{\pi c^2} = \frac{\tau_{\rm Ohm}}{\pi^2}$ the corresponding decay time of the $n = 1$ flat fundamental mode.

Before we turn our discussion of the numerical results it is helpful to view the time evolution of a chosen initial distribution from a complementary point of view. On general grounds $F(t = 0, r)$ as well as its time evolution can be (formally) expanded in a series of the following form:

$$F(t, x) = \Sigma a_n e^{-\frac{c^2 \lambda_n t}{4\pi\sigma R^2}} g_n(x) \tag{4.1}$$

where the summation is extended over all eigenmodes $g_n(x)$ of the corresponding (singular) Sturm-Liouville eigenvalue problem arising from eq. 3.20 and the associated boundary-regularity conditions:

$$Lg_n + \lambda_n e^{-\Phi} g_n = 0 \tag{4.2a}$$

,

$$L = \left(1 - \frac{2Gm(x)}{c^2 x}\right) \frac{\partial^2}{\partial x^2} + \frac{1}{x^2}\left[\frac{2Gm(x)}{c^2} + \frac{4\pi G}{c^2} x^3 \left(\frac{P(x)}{c^2} - \rho\right)\right] \frac{\partial}{\partial x} - \frac{2}{x^2} \tag{4.2b}$$

where in the present case the coefficients in L are determined by the geometrical and hydrodynamical variables of the constant density star solution, $x = \frac{r}{R}$ and R is the areal radius of the star. It follows now from (4.1) that if the eigenvalues are positive and well spaced, then after $t >> \frac{t_{Ohm}}{\lambda_1}$ where λ_1 is the lowest eigenvalue of the above system, then the evolution of the distribution will channel into an exponentially decreasing phase with the dominant contribution in the sum (4.1) coming from the "first" term. Our subsequent described numerical computations exhibits such feature and this property allows to construct numerically the lowest eigenvalue of the above system [30].

In the following numerical calculations we have taken the areal radius to be $R = 10$ km, a constant uniform conductivity $\sigma = 10^{25}$ s^{-1} typical of neutron star values (which implies $\tau_{\rm Ohm} = 4.44\ 10^9$ yrs), and we consider various neutron star masses characterized by different values of the dimensionless compactness ratio: $\epsilon = \frac{2GM}{Rc^2} = 0$, 0.3, 0.4, 0.5, 0.6, 0.740, 0.810, 0.865, and 0.889. The first one corresponds to a flat background space time, the last four are those values used in the numerical plots of ref. [13], while current realistic neutron star models are characterized by ϵ in the range 0.3 to 0.5 [31]. For each value of ϵ, at first we have solved numerically the full relativistic induction equation 3.20, by taking the initial $F(t = 0, r)$ to be equal to the Stoke's function $S(t = 0, r)$ of the corresponding first fundamental decay modes of 3.15 (ie take $n = 1$ and $t = 0$ in 3.15). After performing a long time-integration of eq. 3.20 subject to the conditions cited earlier on, we find that the evolution of $F(t, x)$ channels into an exponentially decaying mode which means, according to 4.1, that the evolution of the initial distribution eventually is described by the first non vanishing term in the series expansion 4.1. This behavior of $F(t, x)$ allows us to determine only the lowest eigenvalue λ_1 of (4.2) from our numerical outputs. Besides the explicit determination of λ_1, our numerical treatment allows us to construct the magnetic field as well. In Fig. 1, we plot as a function of coordinate time t, the magnetic field as perceived by a Killing observer located at the star's pole for the various values of the compactness ratio. Fig.(1) shows that once curvature effects are incorporated and upon ignoring the initial transit time during which the field is in a superposition of various curved eigenmodes, the field follows an exponential decay law (as would have done in the absence of gravity) but now the corresponding e-folding time is longer than the corresponding flat spacetime case. Thus even though we have started with identically prepared systems their evolution is distinct, a distinction traced in the influence of relativistic effects. It should be stressed however that the content of Fig.1 does not by itself provide us with a clear overall picture of the field decay. It rather provides us with a characteristic physical decay time as perceived by a Killing observer situated at the surface of the star and this decay time should not be extrapolated as being the physical decay time over the entire star [32]. In fact, each Killing observer located at some r, will compute a physical e-fold decay time $\tau(r)$ given by: $\tau(r) = Z(r)(\lambda_1)^{-1} = \frac{Z(r)\tau_{Ohm}}{\beta\pi^2}$ and obviously this value changes across the star. (In this formula, we have parametrized λ_1 so that $\beta = 1$ corresponds to flat spacetime). Because of this spatial dependence of $\tau(r)$, in order to get a better insights into the dynamics of the decay, in Fig.2, we have plotted $\lambda_1 = \beta\frac{\pi^2}{\tau_{\rm Ohm}}$, as a function of the compactness ratio ϵ. In the same figure, for comparison purposes, we have plotted the value of the red shift factor at the stars center ($Z_o = e^{\Phi(o)}$) and surface ($Z_s = e^{\Phi(s)}$) respectively. Thus it follows from Fig.(2), that the relativistic corrections to the lowest eigenvalue λ_1, are bounded from above by Z_s while from bellow (almost) by Z_0. It is more instructive however and complements the content of Fig.(2), a plot showing the physical decay times as measured by Killing observers located at the center and at the surface of the star respectively. Fig.(3) stands for such plot, and its content shows that the physical decay time can vary considerably across the star. In fact there are regions around the star's center, where the physical decay time is shorter than the corresponding flat space case and this effect is more pronounced as the compactness ratio increases. In contrast to what occurs in the vicinity of the star's center, in the crust region the physical decay is always larger than the corresponding flat case. Because of this behavior, largely due to gravitational time dilation effect, we assign an overall physical decay time, by averaging the physical e-fold decay at the center and the surface of the star respectively. This amounts to assigning an overral a red shift factor Z for the entire star equal roughly to its value at the middle of the star. With this type of averaging, Figs.(2,3) shows that for small values of the compactness ratio the overall physical decay time is almost identical to the flat space time case. However, as the compactness ratio increases, the relativistic effects become more apparent. For the case of neutrons stars with range in the realistic domain ie ϵ in the range (0.3 , 0.5), and via the averaging procedure outlined above, the overall physical e-fold decay time is $(1.2 - 1.3)$ larger than the corresponding flat case. Although the content of Fig. 1, Fig. 2 and Fig. 3 show the impact of relativistic effects upon the field decay, by themselves they do not offer a clear insight as which (if any) of the two factors, ie red shift or spatial curvature, are responsible for the dominant contribution in the field decay. In order to access their relative importance we solve numerically eq. 3.13a in two extreme cases [33] and show the numerical outputs in Fig.(4). First eq. 3.13a is solved under the assumption $r(l) = l$ and in this approximation the relativistic effects on the decay are solely due to the red shift factor $Z(l)$. In fig.(4) the numerical outputs are indicated by the label: "curved time". In the opposite extreme, we adopt $Z(l) = 1$ in eq. 3.13a and take $r(l)$ as given by the metric corresponding to a constant density star. Thus in this approximation, the only relativistic effect influencing the decay is due to the spatial curvature. The resulting numerical outputs in Fig.(4) are marked by the label "curved space". It follows then clearly from the content of Fig.(4) that for a constant density neutron star, the dominant effect in the field decay is due to the red-shift factor Z, since the corresponding eigenvalues indicated by "curved-time" graph are much closer to the corresponding exact eigenvalue indicated by "curved space-time" in Fig.(4). Moreover, the dominance of the red-shift factor holds through for all values of the compactness ratio ϵ, and increases as ϵ increases.

From the analysis presented so far it is clear that the more compact the star is, the longer is the e-folding time. As a consequence one expects that models of pulsars with soft equation of state to maintain a strong magnetic field for longer period of time than the corresponding models with a stiff equation of state. In turn such slow B field decay implies additional source of heating ie Joule heating and such additional heating, may explain the relatively high temperature observed in old neutron stars. However based on the present analysis, it is rather premature to draw definite conclusions. For instance cooling effects leading to the temporal variation of the conductivity as well as the the detailed structure of the star and its rotation has to be taken into account. Such study is currently under way and we expect to report in a future communication.

V. CONCLUSIONS

The behavior of the surface magnetic fields of neutron stars is a complicated and controversial issue. Many processes are believed to influence its magnitude and its subsequent evolution. Trapping for instance, of the field in the superconducting core is one possibility. The expulsion of the field out of this region is a delicate matter involving many different branches of physics [34]. Another possibility that in principle influences enormously the magnetic properties of neutron stars is related to the accretion processes immediately after the core collapse [35]. Accretion and particularly hypercritical accretion, can submerge the field of the new born neutron star beneath a layer of accreting matter thus in principle producing a delayed switched on mechanism for the pulsar activity [36]. Furthermore according to recent work the neutron star may never turn on as a pulsar [37] if the accretion is hyperctical. Besides the above mechanisms influencing the evolution of neutron star's magnetic fields, many more have been introduced and discussed at length in the current literature. In this work, we have present a limited framework taking into account the effects of space time curvature on the field decay. For the simple neutron star models with a corresponding compactness ratio in the range (0.3 , 0.5), considered in the present work, we have seen an overall increase in the decay time, (1.2 to 1.3) times larger than the flat spacetime value. Although the present work is preliminary and to assess the new effect more work is needed [28], it point towards to the direction that in a strongly gravitating system, effects due to space time curvature should not be neglected.

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APPENDIX A: $(3+1)$ FORM OF MAXWELL'S EQS. ON STATIC SPACETIMES

In this Appendix we shall sketch a derivation of eqs.(2.8) starting from the covariant form of Maxwell's eqs. 2.1. The derivation makes use of the existence of the hypersurface orthogonal timelike Killing field and although all the following computations can be done in a covariant fashion [18,20] for brevity we work explicitly in the coordinate gauge of 2.4. We shall also present formulas required for the derivation of equations in section II.

Starting from the temporal component of the inhomogeneous Maxwell eq. 2.1a, combined with the line element 2.4 and taking into account the fact that $E^j = -F^{jo}e^{\Phi}$ one immediately obtains:

$$D_i E^i = -\frac{4\pi}{c} J^o e^{\Phi} = 4\pi\rho \tag{A1}$$

where we have defined the charge density ρ measured by a Killing observer by: $c\rho = -J^{\mu}U_{\mu}$ and we denote by D the covariant derivative operator associated with the Riemannian metric of the $t = const$ spaces. Due to the fact that the Maxwell tensor $F_{\alpha\beta}$ admits the following easy verifiable decomposition: $F_{\alpha\beta} = U_{\alpha}E_{\beta} - U_{\beta}E_{\alpha} + \epsilon_{\alpha\beta\gamma\delta}U^{\gamma}B^{\delta}$ one gets the following expression for its spatial part F_{ij}: $F_{ij} = \epsilon_{oijl}B^l U^o = \epsilon_{ijl}B^l$. Passing now to the spatial components of 2.1a one gets

$$\frac{\partial E^i}{\partial x^o} - \epsilon^{ijl} D_j(ZB_l) = -\frac{4\pi}{c} ZJ^i \tag{A2}$$

where we have introduced the red shift factor Z via $Z = (-\xi^a \xi_a)^{\frac{1}{2}} = e^{\phi}$ instead of e^{Φ}. On the other hand the second pair of Maxwell's equations 2.1b can be written equivalently as:

$$\frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x^\mu} + \frac{\partial F_{\lambda\mu}}{\partial x^\nu} = 0 \tag{A3}$$

Taking now all the indices to be spatial, and eliminating F_{ij} one gets: $D_i B^i = 0$ The other information encoded in the second pair of Maxwell eqs. can be revealed by considering the following arrangement of the spacetime indices: $(\mu, \nu, \lambda = m, n, x^o)$. For such arrangement one obtains:

$$\frac{\partial(\epsilon_{mnl}B^l)}{\partial x^o} + \frac{\partial(-U_o E_n)}{\partial x^m} + \frac{\partial(U_o E_m)}{\partial x^n} = 0 \tag{A4}$$

from which one easily obtains:

$$\frac{\partial B^l}{\partial x^o} + \epsilon^{lmn} D_m(ZE_n) = 0 \tag{A5}$$

The current conservation eq. $\nabla_\mu J^\mu = 0$ after a trivial rearrangement yields:

$$U^\mu \frac{\partial c\rho}{\partial x^\mu} + D_i J^i + J^i \frac{\partial log Z}{\partial x^i} = 0 \tag{A6}$$

To pass into the equivalent set (2.8-abcd) and eq. 2.9 involving physical orthonormal components, we project all tensors involved onto the natural set of orthonormal vectors (e_i) and one forms (e^i), $\quad (i = 1, 2, 3)$ respectively, associated with the line element 2.7. Thus for instance the electric field E can be written as: $E = E^i \frac{\partial}{\partial x^i} = E^{\hat{i}} e_i$ where $E^{\hat{i}} = h_i E^i$ expresses the relationship between coordinate and frame components of the vector field E and it is understood that no summation is involved over the repeated indices. With the help of the orthonormal components one for instance may rewrite eq. A1 in terms of orthonormal components. Writing $\gamma^{\frac{1}{2}} = h_1 h_2 h_3$ and eliminating the coordinate components in terms of the frame components of E, equation A1 takes the following form:

$$\frac{1}{h_1 h_2 h_3}[\frac{\partial}{\partial x^1}(h_2 h_3 E^{\hat{1}}) + \frac{\partial}{\partial x^2}(h_1 h_3 E^{\hat{2}}) + \frac{\partial}{\partial x^3}(h_1 h_2 E^{\hat{3}})] = \nabla\cdot\vec{E} = 4\pi\rho \tag{A7}$$

where $\nabla\cdot$ stands for the familiar divergence operator expressed in arbitrary orthogonal curvilinear coordinates defined by the line element 2.7. Similarly $D_i B^i = 0$ can be written as $\nabla\cdot\vec{B} = 0$. As far as the other set of Maxwell's eqs. are concerned, one can proceed in a similar manner. For instance starting from A2, one first multiplies the corresponding equation by the scale factor h_i thus leading into:

$$\frac{\partial(E^{\hat{i}})}{\partial x^o} - \frac{\epsilon^{\hat{i}\hat{j}\hat{k}} h_i}{h_1 h_2 h_3} \frac{\partial(h_k B_{\hat{k}} Z)}{\partial x^j} = -\frac{4\pi}{c} J^{\hat{i}} Z \tag{A8}$$

Recalling that the orthonormal components of the *curl* operator of an arbitrary three dimensional differentiable vector field A are given by [38]:

$$(\nabla\times A)^{\hat{i}} = \frac{\epsilon^{\hat{i}\hat{j}\hat{k}} h_i}{h_1 h_2 h_3} \frac{\partial(h_k A_{\hat{k}})}{\partial x^j} \tag{A9}$$

one is lead immediately into eq. 2.8b used in the text. Note also that the action of the gradient operator ∇ acting on scalars is defined via:

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial x^1} + \frac{1}{h_2}\frac{\partial f}{\partial x^2} + \frac{1}{h_3}\frac{\partial f}{\partial x^3} \tag{A10}$$

Also in deriving eqs. (2.8-abc) of the main text we have used the following properties of the unit basis vectors (e_i): $e_{\hat{i}}\cdot e_{\hat{j}} = \delta_{\hat{i}\hat{j}}$, $e_{\hat{i}}\times e_{\hat{j}} = \epsilon_{\hat{i}\hat{j}\hat{k}} e_{\hat{k}}$ and the normalization $\epsilon_{\hat{r}\hat{\theta}\hat{\phi}} = 1$. We may also indicate that for typographical convenience the caret-symbol over frame components of the various tensors has been dropped. In particularly all vector, tensor components appearing anywhere in the main text after eq. 2.7, are frame components.

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[5] A recent discussion concerning possible mechanism leading to the creation of Pulsar's Magnetic field can be found in:

C. Thompson and R. C. Duncan, Astroph. J., 408, 194 (1993).

[6] See for instance: The Cosmic Dynamo, IAU Symp. No. 157 (Eds., F. Krause, K.-H. Rädler, and G. Rüdiger, Kluwer Academic Publishers, (1992)).

[7] For an introduction to dynamo theory see:

F.Krause and K.H. Rädler, Mean-Field Magnetohydrodynamics and Dynamo Theory (Pergamon Press (1980))

[8] For an introduction see:

A. G. Lyne and F. Graham-Smith, Pulsar Astronomy, (Cambridge University Press, (1990)).

[9] S. Sengupta, Astroph. Journal Lett. 479, L133, (1997) and, Astroph. Journal 501, 792, (1998).

[10] For a discussion of the dipole magnetic field on a *Schwarzschild* background see for instance:

V.L. Ginzburg and L.M. Ozernoy, Sov. Phys.-JETP 20, 689, (1964)

J.L. Anderson and J.M. Cohen, Astroph. and Space Sciences, 9, 146, (1970)

I. Wasserman and S.L. Shapiro, Astroph. Journal 265, 1036, (1983)

[11] There are some sensitive issues involved in the use of the terms "influence of general relativistic effects" and "B field decay". It would be perhaps more correctly to write that we are interested to study how certain solutions of Maxwell's equations expressed relative to global Lorentz frames and describing a particular physical situation would appeared once they the same physical problem considered in the presence of non vanishing space time curvature. As far as the term "influence of general relativistic effects" is concerned here rather at the start we are interested in considering the effects of spacetime curvature. Maxwell's eqs. on flat space time can be written in an arbitrary curvilinear system and the issue how a B-decaying field relative to inertial frames would be perceived by in a non inertial observers, naturally arises. The present paper is not intended to analyze such issue. Toward to the end of the paper we shall make a few relevant comments pertinent to that issue.

[12] R.M. Wald, General Relativity (Univ. of Chicago Press (1984))

[13] C. Misner, K. Thorne and J. A. Wheeler, Gravitation (Freeman, (1973)).

[14] A note on our conventions: Maxwell's eqs.(apart from the insertion of the c factor in eq. 2.1a), signature of metric, curvature etc, are the same as that of ref. [12]. In addition in the present work all Greek indices are assumed to be four dimensional, while Latin indices are spatials. The definition and normalization of the four and three dimensional totally antisymmetric Levi-Civita tensor density is the same as that of ref. [12].

[15] H.Weyl: Space-Time-Matter, (Dover 1962).

[16] We shall ignore in the present work possible additional effects of high B-field leading into a conductivity tensor. For simplicity we shall work with a scalar conductivity, leaving the complete treatment to future work.

[17] Such a form of Maxwell's electrodynamics corresponds to a $(3+1)$ formulation. Early work on such approach can be found in:

A.L. Zel'manov, Doclady Akad. Nauk. SSSR, 107, 805, (1956)

F.B. Estabrook and H.D.Wahlquist, J.Math. Phys. 5, 1679, (1964)

G.F.R.Ellis, in Gargese Lectures in Physics, Vol.6 (Ed. E. Schatzman, Gordon and Breach, (1973))

L.D. Landau and E.M. Lifshitz: The Classical Theory of Fields (Perg. Press (1975))

A historical account of those attempts can be found in the ref. [18] bellow.

[18] In addition to the above ref. on (3+1) splitting of curved spacetime electrodynamics, the following reference discusses at great length and generality such a splitting and provides a flexible framework tailored towards to astrophysical applications. Furthermore in that reference the "Absolute space" formalism has been first introduced:

K.S.Thorne and D.Macdonald, Month. Not. R. Astr. Soc. 198, 339, (1982)

In fact we could deduce formulae 2.5 used in the main text by directly appealing to the equations of this work, appropriately restricted to our problem. However for completeness purposes only, we briefly perform a $(3+1)$ splitting of Maxwell's tailored to the static background geometries.

[19] The MHD approximation employed in the present paper is well suited for astrophysical plasmas and particularly for neutron star matter where typically $\sigma \geq 10^{25} sec^{-1}$. To see that let us suppose that $(\vec{E}, \vec{B})$ is a solution of 2.8abc subject to condition $\vec{J} = \sigma\vec{E}$. If by R we denote the typical length scale of the system, then eq. 2.8c implies that in order of magnitude, $\frac{|E|}{|B|} = \frac{R}{cT}$, where T is the dynamical evolution time scale for the E, B fields and in arriving at that estimate we have taken Z to be of order unity. On the other hand $\frac{\partial E}{\partial x^o}\frac{1}{\nabla\times B} = (\frac{R}{cT})^2$ and thus taking into account Ohm's law in eq. 2.8b one obtains $1 = \frac{4\pi\sigma R^2}{c^2}\frac{1}{T} + (\frac{R}{cT})^2$. It follows then from this estimate, if the evolution time scale T is of the order of the Ohmic time, ie $T = \frac{4\pi\sigma R^2}{c^2}$ then for the typical neutron star parameters $\frac{R}{cT} << 1$ and for such situations the displacement current can be neglected from the right hand side of 2.8b implying that the MHD approximation is well justified. Since

essentially in the MHD approximation one neglects the electromagnetic radiation, that point will allows us further on to join the interior solutions with static exterior dipole field.

[20] Black holes: The membrane Paradigm, (Eds K. Thorne, R.H. Price and D.A. Macdonald, Yale Univ. Press (1986)). In that collection of articles the original "Absolute Space" approach of curved electrodynamics of ref. [18] has been elaborated further and extensively applied to black hole spacetimes. We have been charmed by the practical usefulness of this approach and in that spirit we have written Maxwell's equations in the form 2.8. For the present case the "absolute space" is identified by the three dimensional spacelike sections perpendicular to the Killing field. We should stress however that there are many more advantages of the "Absolute space" formulation of curved spacetime electrodynamics than its mere practical usefulness, and the interested reader is refered to the above volume for more detailed applications.

[21] Actually even for a time dependented σ, the toroidal and poloidal component evolve independently of each other. In the present work we shall use in section III a uniform conductivity, however as we go along we shall point out implications on the evolution of the magnetic field components due to a conductivity characterized by an arbitrary spacetime dependence.

[22] In the case of an arbitrary l-pole field one recovers an identical equation as the above. The sole exception is that in the factor of 2, in last term of the right hand-side is replaced by $l(l+1)$ respectively.

[23] See for instance:

Y. Sang and G. Chanmugam, Astrophys. Journal, 363, L61, (1987)

[24] As far as we are aware the so called "Stoke's function" has been introduced as a convenient parametrization of the poloidal fields in:

P.M. Morse and H. Feshbach: Methods of Theoretical Physics (McGraw-Hill, N.York (1953))

[25] Strictly speaking the interior field for both flat and curved space case, should be joined with a radiating into empty space solution of Maxwell's eqs. However due to the MHD approximations and due to the long times evolved in the decay of the interior field typically and to a good approximation one considers the exterior dipole field to consist of a sequence of quasi static dipole solutions. Thus essentially we take the exterior dipole magnetic moment to be a slow varying function of time and that approximation allows us to perform the C^o mashing of the $\vec{B}$ across the star's surface.

[26] Exact solutions of this equation have been obtained for example in ref. [23] above. The sequence of eigenvalues have been known long ago, see for instance:

H.Lamb Phil. Trans.Roy.Soc. Lond. 174, 519, (1883)

[27] Early attempts to take into account the influence of the electromagnetic stresses on the structure of neutron stars can be found in:

G.Dautcourt and K.Fritze, Astron. Nachr., 295H, 211, (1971).

More recently a large scale numerical computations of the structure of rotating neutron stars taking into account the effects of the Maxwell field has been performed by:

M. Bocquet, S. Bonazolla, E. Gourgoulhon and J. Novak, Astron. and Astroph. 301, 757, (1995).

According to the results of this study and under the assumption that the B field is purely poloidal, the effects of the Maxwell field does not yield appreciably different neutron stars models than the conventional models for field strengths $B < 10^{13}G$. For larger fields they report differences than the conventional models but issues related to stability of such models have not been addressed yet.

[28] D.Page, U. Geppert and T. Zannias, (Subm. A.A. 2000)

[29] W.H. Press, B.P. Flanery, S.A. Teukolsky and W.T. Vetterling: Numerical Recipes (C.U.P. (1986)) also; Web site: http://www.nr.com

[30] It should be stressed however that knowledge of all eigenvalues yields information regarding the behavior of the magnetic field before the exponentially decreasing phase is reached. Accordingly construction of the other eigenvalues is a worthwhile project.

[31] M. Prakash, 'Neutron Stars', in Nuclear and Particle Astrophysics, Eds. J. G. Hirsch & D. Page (Cambridge University Press (1998)).

[32] It should be stressed here that even though we are drawing physical conclusions based on the magnetic field measured by Killing observers, the influence of the curvature can be seeing and described in a coordinate and observer free manner. One for instance may consider the field invariant $F^{\alpha\beta}F_{\alpha\beta} = 2[(\vec{B})^2 - (\vec{E})^2]$ and examine its properties on a flat and curved spacetime. Within the MHD approximation, the $\vec{E}$ field can be computed via Ohm's law and thus the right handside of this field invariant is expressible in terms of the corresponding Stokes functions. In fact for our case since the $\vec{E}$ and current $\vec{J}$, are purely toroidal, they vanish on the poles of the star, hence $[F^{\alpha\beta}F_{\alpha\beta}(t, R, \theta = 0)]^{\frac{1}{2}} \simeq |\vec{B}(t, R, \theta = 0)|$. Thus Fig.(1) also supplies information regarding the behavior of this field invariant for the curved-flat space respectively. Accordingly the "flat value" of $F^{\alpha\beta}F_{\alpha\beta}(t, R, \theta = 0)$ decays more rapidly than its curved counterpart. We have chosen to indicate the effect in terms of the $\vec{B}$ since the latter is directly related to terrestrial observations.

[33] In our numerical integration of (3.13a) for the cases $Z = 1$ and $r(l) = l$ respectively, we have used as boundary condition the expression (3.18). Although this is not entirely correct, numerically we have found that the content of Fig.(4) is rather insensitive to small changes in the surface boundary condition.

[34] It is rather hard, due to the rapid evolution of the field to give an update reference on the subject. However the following text gives a concise introduction of the basic

principles involved in the Physics of the neutron stars:

S.L. Shapiro and S.A. Teukolsky: Black Holes, White Dwarfs and Neutron Stars, The Physics of Compact Objects, (Wiley-Interscience-Pub. (1983))

In addition an overview the various possibilities regarding the behavior of the core magnetic fields can be found in:

J. Sauls: in "Timing of Neutron Stars" (eds. H.Ogelman and E.P.J. van der Heuvel, Kluwer (1989))

[35] The idea that the core collapse and supernovae explosion may also accompanied by a heavy accretion has been discussed long ago in the following refs.

S.Colgate Astrop. Journal, 163, 221, (1971)

Y.B. Zeldovich, L.N. Ivanova and D.K. Nadezhin, Sov. Astron., 16, 209, (1972)

[36] A. Muslimov and D. Page, Astrop. Journal, L77, 440, (1995), idid 458, 347, (1995)

[37] U. Geppert, D. Page and T. Zannias, Astron. and Astroph. 345, 847, (1999) and Mem.S.A.It.,69,4 (1998)

[38] For a review of a few elementary properties the vector calculus in orthogonal curvilinear coordinates consult ref. [20] above as well as:

Fundamental Formulas in Physics (Ed. D.H. Menzel Vol. II Dover (1960))

FIGURES

FIG. 1. Figure 1

Dipolar field decay for a uniform density star in curved and flat space-time. The horizontal axis represents (coordinate) time in units of the flat space-time ohmic decay time $\tau_{ohm} \equiv 4\pi\sigma R^2/c^2$ and the vertical axis shows the value of $B/B_0 = B(t, r = R, \theta = 0)/B(t = 0, r = R, \theta = 0)$. All models have the same areal radius ($R = 10$ km) and a constant uniform conductivity ($\sigma = 10^{25}$ s^{-1}) typical of neutron star values (which implies $\tau_{ohm} = 4.44\ 10^9$ yrs). The values of the compactness ratio $2GM/Rc^2$ is indicated on each plot. The initial field profile is taken as the $n = 1$ eigenmode for the flat space-time, Eq.3.15, in all cases. The graphs show quite clearly the exponential decay in flat space-time, with $\tau = \tau_{ohm}/\pi^2$ within numerical accuracy, while, as expected, in curved space-time the decay initially deviates from an exponential law but rapidly converges toward the corresponding fundamental mode.

FIG. 2. Figure 2

The horizontal axis stands for the dimensionless comactenes ratio $2GM/Rc^2$, while the vertical axis corresponds to the values of β. The graphs marked by e^{Φ_o} , e^{Φ_s} stand for the value of the red shift factor at the center and surface of the star respectively.

FIG. 3. Figure 3

The horizontal axis stands for the dimensionless compactenes ratio $2GM/Rc^2$, while the vertical axis corresponds to the ratio of the physical decay time τ_{ph} over the corresponding flat value τ_{fl}. The graphs marked as "surface", "center" respectively, represents $\frac{\tau(R)_{ph}}{\tau_{fl}} = \frac{Z_s}{\beta}$, $\frac{\tau(0)_{ph}}{\tau_{fl}} = \frac{Z_o}{\beta}$ while the corresponding horizontal line through (0 , 1) stands for the flat case.

FIG. 4. Figure 4

The horizontal axis stands for the dimensionless comactenes ratio $2GM/Rc^2$, while the vertical axis corresponds to the values of β. The graphs marked as "curved space","curved time" provides the eigenvalues corresponding the the case where $Z = 1$ and $r(l) = l$ respectively, as explained in the text, while the graph marked as "curved space time" corresponds to the exact equation.