

# Negative Reynolds stress generation by accretion disc convection

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## ABSTRACT

The phenomenon of negative viscosity- $\alpha$  in convectively unstable Keplerian accretion discs is discussed. The convection is considered as a random flow with an axisymmetric mesoscale pattern. Its correlation tensor is computed with a time-averaging procedure using Kley’s 2D hydrocode. There is a distinct anisotropy between the turbulence intensities in the radial and azimuthal directions, i.e. the radial velocity rms dominates the azimuthal one. As a consequence, an extra term in the expression for the turbulent transport of angular momentum appears which does not vanish for rigid rotation (‘ $\Lambda$ -effect’). It is negative (‘inwards transport’) and even seems to dominate the positive contribution of the eddy viscosity representing outwards transport of angular momentum. For a turbulence model close to that of the mixing-length theory, the rotational influence on the anisotropy of the turbulence intensities,  $\langle u_R'^2 \rangle - \langle u_\phi'^2 \rangle$ , and the covariance  $\langle u_R' u_\phi' \rangle$  – representing the angular momentum transport – is computed and compared with the accretion disc simulations. Indeed, the negative angular momentum transport can be explained with the observed dominance of the radial turbulence intensity. If, on the other hand, in turbulence fields the azimuthal intensity would dominate or the turbulence is even isotropic, then we always find a positive transport of the angular momentum.

**Key words:** accretion, accretion discs – convection – planetary systems: protoplanetary discs.

## 1 MOTIVATION

In accretion discs, the inward transport of matter is always accompanied by an outward transport of angular momentum. The latter is imagined as being due to small-scale chaotic flow (‘turbulence’) with properties described by its correlation tensor

$$Q_{ij} = \langle u_i'(\mathbf{x}, t) u_j'(\mathbf{x}, t) \rangle. \quad (1)$$

If cylindrical coordinates  $(R, z, \phi)$  are used, then the  $R$ – $\phi$  component of (1) describes the desired radial transport of angular momentum. As is well known, an isotropic turbulence field only transports angular momentum if the angular velocity is non-uniform. Then one can write

$$Q_{R\phi} = -\nu_T R \frac{d\Omega}{dR}. \quad (2)$$

Within the magnetohydrodynamic (MHD) regime, the Maxwell stress also has to be considered (cf. Abramowicz, Brandenburg & Lasota 1996). As is custom in the accretion theory, the eddy viscosity is parametrized by

$$\nu_T \propto H^2 \Omega, \quad (3)$$

with  $\Omega$  as the local rotation rate and  $H$  the half-thickness of the

disc. For a Kepler flow, therefore, one finds

$$Q_{R\phi} = \alpha_{SS} c_{ac}^2 \quad (4)$$

with  $c_{ac}$  as the speed of sound and with the viscosity- $\alpha$  after Shakura & Sunyaev (1973) as the free parameter without any dimension. The quantity (4) is positive definite; it can never be negative. This argumentation complies with the finding of Krause & Rüdiger (1974a) that the eddy viscosity for stationary and isotropic turbulence,

$$\nu_T = \frac{8}{15} \iint \frac{\nu^3 q k^8}{(\omega^2 + \nu^2 k^4)^2} d\mathbf{k} d\omega \quad (5)$$

( $q$  spectral function,  $k$  wavenumber,  $\omega$  frequency), is positive definite. It even remains finite in the case of vanishing microviscosity,  $\nu \rightarrow 0$ . Much more complicated is the situation for anisotropic turbulence. In the two-dimensional limit, for example, the eddy viscosity can be negative but the sum of microscopic and turbulence viscosity always remains positive (Krause & Rüdiger 1974b).

The question of whether eddy viscosities can be found as negative by theory or observations was first formulated by Starr (1968). With positive eddy viscosity and for Kepler rotation law the correlation (2) is always positive and the angular momentum transport is always outwards. However, the angular momentum transport for *anisotropic* turbulence fields cannot be described by

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the simple Boussinesq approximation in equation (2). The correlation tensor for rotating anisotropic turbulences does *not* vanish for rigid rotation (cf. Kippenhahn 1963; Gough 1978). The simplest modification of equation (2) is the inclusion of a non-shear term, i.e.

$$Q_{R\phi} = -\nu_{\text{T}} R \frac{\partial \Omega}{\partial R} + \Lambda_V \Omega \quad (6)$$

(cf. Rüdiger 1989). The  $\Lambda$  term can be understood as a hydrodynamical analogue to the  $\alpha$ -effect of MHD. It is the zero-order term in a Taylor series development after derivatives of the basic rotation  $\Omega$ .

There is a known tensorial formulation of the  $\Lambda$  term in equation (6). Any anisotropic turbulence possesses a characteristic direction, say  $\mathbf{g}$ . Then the linear influence of the basic rotation on the correlation tensor can be described by the expression

$$Q_{ij} = -\Lambda(\epsilon_{ipk} g_j + \epsilon_{jpk} g_i) g_p \Omega_k, \quad (7)$$

which for its  $R$ - $\phi$  component results in

$$Q_{R\phi} = \Lambda(g_R^2 - g_\phi^2)\Omega. \quad (8)$$

Note that the vertical, i.e. the  $z$ -component, of the stratification vector  $\mathbf{g}$  never contributes, so the vertical stratification of accretion discs can never produce any radial angular momentum transport. If, on the other hand, any anisotropy exists in the radial direction (such as a pressure gradient or curvature), then a finite value of the angular momentum transport is an unavoidable consequence. The sign of equation (8) is the remaining open question.

The sign results from the type of anisotropy as we shall show in the following by means of a quasi-linear approximation. The turbulent flow pattern is split into  $\mathbf{u}' = \mathbf{u}^{(0)} + \mathbf{u}^{(1)}$ , where  $\mathbf{u}^{(0)}$  is a forced turbulence which is modified to  $\mathbf{u}^{(1)}$  by the action of rotation. It fulfils the equation

$$\frac{\partial u_i^{(1)}}{\partial t} - \nu \Delta u_i^{(1)} + 2\epsilon_{ijk} \Omega_j u_k^{(0)} = -\frac{1}{\rho} \frac{\partial p^{(1)}}{\partial x_i} - \bar{u}_{l,m} x_m u_{i,l}^{(0)} - u_l^{(0)} \bar{u}_{i,l}, \quad (9)$$

in first order of both rotation and shear. Mass conservation requires  $\text{div } \mathbf{u}^{(0)} = \text{div } \mathbf{u}^{(1)} = 0$ . The resulting correlation, also taken in first order, is

$$Q_{ij} = \langle u_i^{(0)}(\mathbf{x}, t) u_j^{(1)}(\mathbf{x}, t) + u_j^{(0)}(\mathbf{x}, t) u_i^{(1)}(\mathbf{x}, t) \rangle, \quad (10)$$

whose rotation-created part (odd in  $\Omega$ ) may be written as  $Q_{ij} = \Lambda_{ijk} \Omega_k$ . Note that equation (9) is *not* an equation with constant coefficients, so a modified Fourier procedure must be applied. The result is provided by Rüdiger (1989) and reads

$$\Lambda_{ijk} = 2\nu \iint \left\{ \epsilon_{ilm} \hat{Q}_{lj}^{(0)} + \epsilon_{jlm} \hat{Q}_{il}^{(0)} \right\} \frac{k_n k_k \mathbf{d}\mathbf{k} \, d\omega}{\omega^2 + \nu^2 k^2}. \quad (11)$$

Here only the linear expression in  $\Omega$  is given with which the basic role of the anisotropies within the turbulence field can already be demonstrated. The spectral tensor of the prescribed ‘original’ turbulence field is  $\hat{Q}_{ij}^{(0)}$ , defined after

$$Q_{ij}^{(0)}(\mathbf{x}, t) = \iint \hat{Q}_{ij}^{(0)}(\mathbf{k}, \omega) e^{i(\mathbf{k}\mathbf{x} - \omega t)} \mathbf{d}\mathbf{k} \, d\omega. \quad (12)$$

The simplest example is a homogeneous and isotropic (in 3D) field of turbulence with its spectral tensor  $\hat{Q}_{ij}^{(0)} = q(\mathbf{k}, \omega) \times (k^2 \delta_{ij} - k_i k_j)$ . Such a field, however, does not produce any  $\Lambda$ -effect. More important is anisotropic turbulence. A turbulence with

no intensity along the direction  $\mathbf{g}$ , i.e.  $g_i Q_{ij}^{(g)} = 0$ , is described by

$$\hat{Q}_{ij}^{(g)} = q^{(g)} [\tilde{k}^2 (\delta_{ij} - g_i g_j) - \tilde{k}_i \tilde{k}_j], \quad (13)$$

with  $\tilde{\mathbf{k}} = \mathbf{k} - (\mathbf{g} \cdot \mathbf{k}) \mathbf{g}$  and  $q > 0$ . Therefore, with

$$\hat{Q}_{ij}^{(0)} = q(k^2 \delta_{ij} - k_i k_j) + \hat{Q}_{ij}^{(g)} \quad (14)$$

we have a turbulence with a dominating *azimuthal* intensity

$$\langle u_\phi'^2 \rangle - \langle u_R'^2 \rangle = \iint q^{(g)} k_z^2 \mathbf{d}\mathbf{k} \, d\omega > 0, \quad (15)$$

if  $\mathbf{g}$  describes the radial unit vector. If the latter is replaced by the unit vector  $\mathbf{e}$  in azimuthal direction, a turbulence field is described with

$$\langle u_R'^2 \rangle - \langle u_\phi'^2 \rangle = \iint q^{(e)} k_z^2 \mathbf{d}\mathbf{k} \, d\omega > 0, \quad (16)$$

i.e. the *radial* intensity dominates. In the following we, therefore, have to compute the Reynolds stress (8) with the general turbulence field

$$\hat{Q}_{ij}^{(0)} = q(k^2 \delta_{ij} - k_i k_j) + \hat{Q}_{ij}^{(g)} + \hat{Q}_{ij}^{(e)}, \quad (17)$$

i.e.

$$\langle u_R'^2 \rangle - \langle u_\phi'^2 \rangle = \iint [q^{(e)} - q^{(g)}] k_z^2 \mathbf{d}\mathbf{k} \, d\omega. \quad (18)$$

From (11) one finds

$$Q_{R\phi}^\Lambda = 2\Omega \iint \left\{ k_z^2 [\hat{Q}_{\phi\phi}^{(0)} - \hat{Q}_{RR}^{(0)}] + k_R k_z \hat{Q}_{Rz}^{(0)} - k_\phi k_z \hat{Q}_{\phi z}^{(0)} \right\} \times \frac{\nu \mathbf{d}\mathbf{k} \, d\omega}{\omega^2 + \nu^2 k^4}. \quad (19)$$

Only the anisotropic parts of the turbulence are of interest, i.e.

$$Q_{R\phi}^\Lambda = 2\Omega \iint \left[ (k_z^2 + k_\phi^2) q^{(g)} - (k_z^2 + k_R^2) q^{(e)} \right] \frac{\nu k_z^2 \mathbf{d}\mathbf{k} \, d\omega}{\omega^2 + \nu^2 k^4}. \quad (20)$$

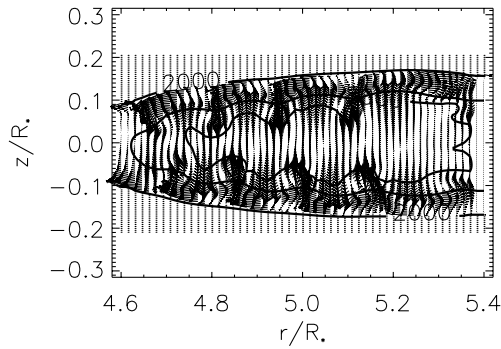
Obviously, any sign is possible of this quantity. For  $q^{(e)} = 0$  we find positive values for  $Q_{R\phi}$  and  $\langle u_\phi'^2 \rangle > \langle u_R'^2 \rangle$  from (18). We have the opposite situation for  $q^{(g)} = 0$ , for which the  $\Lambda$ -effect is negative after (20) and the turbulence is radially dominated after (18).

A rather clear result follows for a turbulence field which can be described by axisymmetric rolls. We have to put  $q = q^{(g)} = 0$  in this case, so that

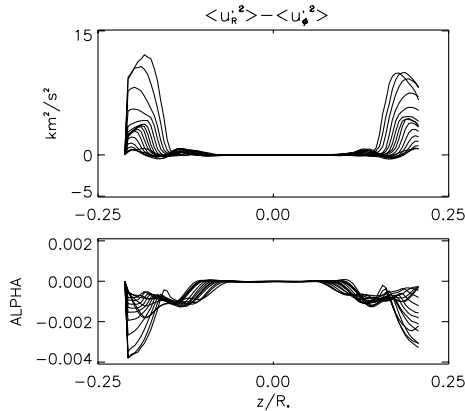
$$Q_{R\phi} = -2\Omega \iint \frac{\nu k_z^2 (k_R^2 + k_z^2)}{\omega^2 + \nu^2 k^4} q^{(e)} \mathbf{d}\mathbf{k} \, d\omega \quad (21)$$

results which is negative-definite. Two-dimensional east-west rolls are always transporting the angular momentum inwards (cf. Rüdiger 1989). However, all possible deviations from this simple flow pattern tend to change the sign of  $Q_{R\phi}$  to positive values.

Whether or not even the *total angular momentum* can be negative depends on the intensity of the isotropic part of the turbulence field. If the eddy viscosity is always positive then with  $\partial\Omega/\partial r < 0$  the second term in (6) is always positive so that the total  $Q_{R\phi}$  is always positive for positive  $\Lambda_V$ . The total angular momentum transport can only become negative for sufficiently negative  $\Lambda_V$ . All in all, our approximations accepted, a necessary condition for negative angular momentum transport is *the dominance of the radial turbulence intensity over the azimuthal*



**Figure 1.** The flow pattern for an accretion disc model with  $\alpha_{SS} = 0.01$  and  $\dot{M} = 3 \times 10^{-5} M_{\odot} \text{yr}^{-1}$ . Temperature contours are given for 2000, 5000 and 8000 K to demonstrate the convective character of the flow. The computational domain covers the radius between  $4R_{*}$  and  $6R_{*}$ .



**Figure 2.** The basic correlations after a time-averaging for the model of Fig. 1 at many different radial positions. Top panel:  $\langle u_r'^2 \rangle - \langle u_{\phi}'^2 \rangle$ , representing the anisotropy of the convective flow along the  $z$ -coordinate. The radial turbulence intensity dominates the azimuthal one. Bottom panel: the normalized angular momentum flow  $Q_{R\phi}$  by the convection proves to be negative where the turbulence field is anisotropic.

one. In Section 3 we shall demonstrate this finding with a special numerical model of convective accretion discs.

In reality, the coefficient quantities in equation (6) are functions of the rotation rate  $\Omega$  in the form of its Coriolis number

$$\Omega^* = 2\tau_{\text{conv}}\Omega. \quad (22)$$

For fast rotation this number exceeds unity, and it is not obvious whether equation(6) also persists in this case (which seems to be the realistic one for accretion discs). Thus, we also have to compute the quantities  $\Lambda_V$  and  $\nu_T$  in equation (6) for

$$\tau_{\text{rot}} < \tau_{\text{conv}}. \quad (23)$$

The results for such computations are summarized at the end of Section 3. One can take from the numbers given in Figs 1 and 2 that the Coriolis number of the presented convective pattern indeed varies between 1 and 10.

## 2 NUMERICAL SIMULATIONS

Ruden, Papaloizou & Lin (1988) have started to study axisymmetric perturbations of a thin, convectively unstable but inviscid Keplerian disc. In two dimensions, the eddy viscosity vanishes for inviscid flows (Krause & Rüdiger 1974b), hence finite

viscosity-alpha in this case can only be due to highly non-linear secondary interactions.

The non-axisymmetric case is considered in detail by Ryu & Goodman (1992), where negative correlations  $Q_{R\phi}$  are obtained for convection in Keplerian discs. They also find a non-zero angular momentum flux for non-axisymmetric modes, but directed inwards, i.e. towards the direction of increasing angular velocity. As for three dimensions, the eddy viscosity, even in the limit  $\nu \rightarrow 0$ , proves to be positive; the negative sign of the angular momentum flux can only be understood by means of an extra part aside from the eddy viscosity in the Boussinesq relation (2). Cabot & Pollack (1992) provide detailed computations of the covariance  $Q_{R\phi}$ . For large rotation rates it becomes negative near the walls (their fig. 5). In this paper we also find an indication for anisotropy of the turbulence intensity, i.e.

$$\langle u_{\phi}'^2 \rangle < \langle u_r'^2 \rangle, \quad (24)$$

and an increase of the correlation length in the azimuthal direction due to the action of shearing. Thus after equation (6), negative values for  $Q_{R\phi}$  can result in large  $\langle u_r'^2 \rangle$ .

Kley, Papaloizou & Lin (1993), with their 2D simulations of convection in a medium with a rather large eddy viscosity (equation 3), also obtain a flow system which appears to yield an inward transport of angular momentum (cf. Goldman & Wandel 1995). Furthermore, negative values for the covariance  $Q_{R\phi}$  also appear in the 3D simulations of an inviscid fluid of Stone & Balbus (1996), probing the role of vertical convective motions in providing angular momentum transport in a Keplerian disc. Of particular relevance for our discussion is a simulation for rigid rotation ( $\partial\Omega/\partial R = 0$ ) which yields small but negative radial angular momentum transport.

Igumenshchev, Abramowicz & Narayan (2000) present 3D hydrodynamic simulations of convective advection-dominated accretion flows. They also report a strong tendency of the eddies toward axisymmetry and, simultaneously, an inward transport of angular momentum (cf. their Fig. 4).

A strong tendency to axisymmetry due to differential rotation has also been found by Cabot (1996). The resulting convection is antisymmetric with respect to the equator. Nevertheless, by definition, the correlation  $Q_{R\phi}$  is a symmetric function with respect to the equator. The average in Cabot's paper is over  $\phi$  and  $R$ ; the amplitude of  $\alpha_{SS}$  is of the order of  $10^{-3}$ .

## 3 RESULTS

It is not easy to probe our findings about the turbulence anisotropy and the sign of the resulting  $Q_{R\phi}$  with a detailed disc model. Very often the disc instability has a magnetohydrodynamical background. However, there is one exception. A model has been developed by Kley et al. (1993) and Klahr, Henning & Kley (1999) to simulate the convection in cool discs. The model works with a stress-strain relation due to an instability which is not important here but which can be described by an internal viscosity parameter  $\alpha_{SS}$ . In order to remain within the hydrodynamic regime, *cold* discs are in particular considered. This is the reason why the hydrocode presented in Kley et al. (1993) has been used to probe the *convective flow* pattern for angular momentum transport in the light of the above considerations. As the code does not perform non-axisymmetric variations, our numerical simulations only concern ring-like disturbances.

The re-discussion of the theory of cold accretion discs (like the

protoplanetary discs) including non-axisymmetric instabilities by Klahr et al. (1999) again reveals the strong tendency of the flow to form axisymmetric east–west rolls. On the other hand, the present-day renaissance of ideas about the linear and non-linear instability of non-magnetic Kepler flows (cf. Richard & Zahn 1999; Huré, Richard & Zahn 2001; see, however, Balbus, Hawley & Stone 1996; Brandenburg & Dintrans 2001) underlines the importance of the relations between turbulence anisotropy and the sign of the angular momentum transport given in the present paper.

The radial transport of angular momentum follows from

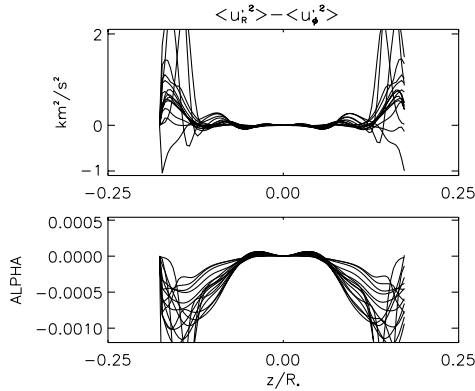
$$Q_{R\phi} = \langle u_R u_\phi \rangle - \langle u_R \rangle \langle u_\phi \rangle, \quad (25)$$

if density fluctuations are generally ignored. Again, an alpha-viscosity expression is introduced in the sense of the normalized angular momentum transport due to convection, i.e.

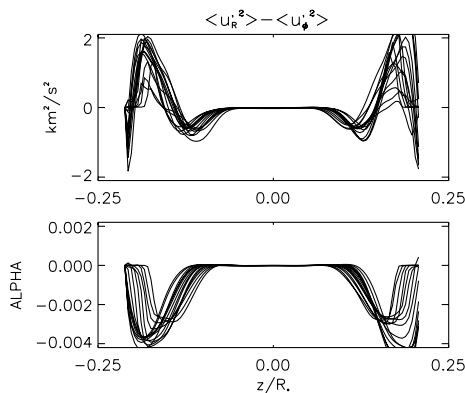
$$\alpha_{\text{conv}} = Q_{R\phi}/c_{\text{ac}}^2, \quad (26)$$

which is given in Figs 2–4 for various disc models.

The average procedure is only concerned with time. As a result of the symmetry of the flow pattern, there is no problem to also involve the vertical direction in the average procedure. The convective flow pattern is shown in Fig. 1. In the temporal average the flow exhibits an equatorial antisymmetry with cells crossing the equator. The vertical profiles shown in Fig. 2 reveal the flow pattern as anisotropic in the  $R$ – $\phi$  plane, too. The turbulence intensity in the radial direction dominates the turbulence intensity in the azimuthal direction. Therefore, after equation (21) we expect a negative  $\Lambda$  effect, and if it is strong enough we then expect an *inwardly* directed flow of angular momentum. Indeed, as suggested



**Figure 3.** The same as in Fig. 2 but for  $\alpha_{\text{SS}} = 0.05$ .



**Figure 4.** The same as in Fig. 2 but for  $\alpha_{\text{SS}} = 0.005$ .

by the quasi-linear theory presented in Section 1, the relation

$$\langle u_\phi'^2 \rangle - \langle u_R'^2 \rangle \langle u_\phi' u_R' \rangle > 0 \quad (27)$$

appears to be fulfilled. If, however, the anisotropy becomes smaller and smaller – such as indicated in Fig. 3 for a large value of  $\alpha_{\text{SS}}$  – then the amplitude of the (negative) viscosity-alpha also becomes smaller and smaller, as also indicated in Fig. 3.

The viscosity- $\alpha$  in the Kley-code here only represents the amplitude of the internal viscosity in the simulation. Hence, the smaller the  $\alpha_{\text{SS}}$  the more realistic is the model. Fig. 4 presents the results for a model with a small  $\alpha_{\text{SS}}$ . We see that the results hardly differ from those in Fig. 2, i.e. the simultaneous existence of radially dominated turbulence and an inwards transport of angular momentum occurs again.

Equation (21) only holds for slow rotation,  $\Omega^* < 1$ . It has been underlined at the end of Section 1 that the case of rapid rotation,  $\Omega^* \geq 1$ , must also be considered. In that case both the quantities  $\Lambda$  and  $\nu_T$  depend on the rotation rate in the form of the Coriolis number (10) and we write

$$Q_{R\phi} = (\langle u_\phi'^2 \rangle - \langle u_R'^2 \rangle) \lambda(\Omega^*) - \langle u_R'^2 \rangle \frac{\partial \log \Omega}{\partial \log R} \nu(\Omega^*), \quad (28)$$

where  $\lambda$  and  $\nu$  are dimensionless functions of  $\Omega^*$  which describe the rotation-dependence of the eddy viscosity and the  $\Lambda$  effect. Again, as can be seen in equation (28) the latter stems from the anisotropies in the turbulence field.

In order to provide the functions  $\lambda$  and  $\nu$  we have to determine the rotational influence on a given anisotropic turbulence field (Kitchatinov & Rüdiger 1993; Kitchatinov, Pipin & Rüdiger 1994). In our current notation the results can be written as

$$\lambda = \frac{3}{4\Omega^{*3}} \left( \frac{3 + \Omega^{*2}}{1 + \Omega^{*2}} + \frac{\Omega^{*2} - 3}{\Omega^{*2}} \arctan \Omega^* \right) \quad (29)$$

and

$$\nu = \frac{3}{64\Omega^{*3}} \left( \Omega^{*2} - 21 - \frac{8\Omega^{*2}}{1 + \Omega^{*2}} + \frac{21 + 14\Omega^{*2} + \Omega^{*4}}{\Omega^{*2}} \arctan \Omega^* \right). \quad (30)$$

In order to simplify the expressions a mixing-length approximation has been applied which leads to the definition of a correlation time  $\tau_{\text{corr}}$ . In the limit of slow rotation one finds

$$\lambda \approx \frac{4}{5} \Omega^*, \quad \nu \approx \frac{2}{5} \Omega^*, \quad (31)$$

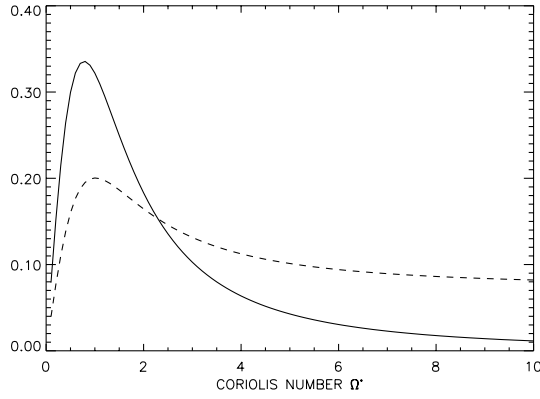
so that the simplest expression for a turbulence field subject to Kepler rotation reads

$$Q_{R\phi} = \frac{\Omega^*}{5} (4\langle u_\phi'^2 \rangle - \langle u_R'^2 \rangle). \quad (32)$$

The turbulent angular momentum transport, therefore, is positive for dominating azimuthal turbulence intensity and it is negative for dominating radial turbulence intensity. If only orders of magnitudes are considered, expressions (26) and (28) result in the relation

$$\alpha_{\text{conv}} \approx \lambda \text{Ma}^2, \quad (33)$$

with the Mach number  $\text{Ma} \approx \sqrt{\langle u_R'^2 \rangle}/c_{\text{ac}}$ .



**Figure 5.** The rotation dependence of the dimensionless coefficients  $\lambda$  (solid line) and  $\nu$  (dashed line) in the turbulent angular momentum transport (equations 29 and 30).

In Fig. 5 the profiles of  $\lambda(\Omega^*)$  and  $\nu(\Omega^*)$  are given. For an  $\Omega^*$  value of about 5 we find  $\lambda$  to be of the order of 0.05 so that (33) reads  $\alpha_{\text{conv}} \approx 0.05 \text{ Ma}^2$ . The Mach number in our simulations is slightly larger than 0.2, so that the models are consistent for  $\alpha_{\text{conv}}$  equal to the order of  $10^{-3}$ , as is indeed the case in all our simulations.

#### 4 ROTATION AND ANISOTROPY

It is true that the turbulence intensities  $\langle u_R'^2 \rangle$  and  $\langle u_\phi'^2 \rangle$  presented in Figs 2–4 are not identical with the turbulence intensities  $\langle u_R'^2 \rangle$  and  $\langle u_\phi'^2 \rangle$  in equation (16). The latter are taken without rotation, whereas the values in Figs 2–4 are subject to the basic rotation. For a comparison of both quantities one has to compute the general influence of a basic rotation to a given original turbulence field with the spectral tensor  $\hat{Q}_{ij}^{(0)}$ . In Rüdiger (1989) the spectral tensor  $\hat{Q}_{ij}$  for the rotating turbulence field is given as

$$\hat{Q}_{ij} = \frac{\omega^2 + \nu_1^2 k^4}{NN^*} \left\{ \left[ \omega^2 + \nu_1^2 k^4 - \left( \frac{2k\Omega}{k} \right)^2 \right] \hat{Q}_{ij}^{(0)} + \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) \left( \frac{2k\Omega}{k} \right)^2 \hat{Q}_{ll}^{(0)} \right\} \quad (34)$$

with the denominator

$$N = (-i\omega + \nu_1 k^2)^2 + \left( \frac{2k\Omega}{k} \right)^2 \quad (35)$$

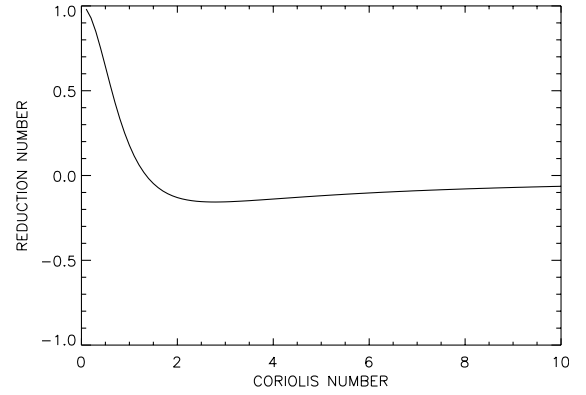
and  $N^*$  its complex conjugate. Here  $\omega$  is the frequency and  $k$  is the wavenumber in the power spectrum of the turbulence. Provided the turbulence is of mixing-length type (cf. Kitchatinov 1987), the resulting anisotropy in the rotating turbulence field is

$$\langle u_R'^2 \rangle - \langle u_\phi'^2 \rangle = A(\Omega^*) (\langle u_R^{(0)2} \rangle - \langle u_\phi^{(0)2} \rangle) \quad (36)$$

with the ‘transformation’ factor

$$A = \frac{1}{8\Omega^{*4}} \left( \frac{15 - 20\Omega^{*2} - 11\Omega^{*4}}{1 + \Omega^{*2}} + \frac{-15 + 30\Omega^{*2} - 3\Omega^{*4}}{\Omega^*} \arctan \Omega^* \right).$$

A series expansion after small  $\Omega^*$  reads  $A \approx 1 - 13\Omega^{*2}/7$  (Fig. 6). Obviously, the basic rotation strongly suppresses any originally



**Figure 6.** The rotation dependence of the transformation factor  $A$  after equation (37) describing the behaviour of the radial–azimuthal anisotropy in the turbulence field for various rotation rates.

existing anisotropy in the turbulence intensities. The observed anisotropy in the Figs 2–4 is thus the reduced representation of the ‘true’ anisotropy of the turbulence field. The fact that for large values of the Coriolis number  $\Omega^*$ , i.e. for long-living convection, the transformation factor  $A$  becomes smaller and smaller and even changes its sign is a surprising result. Canuto, Minotti & Schilling (1994) demonstrate for their rather general model how the global rotation influences the formation of anisotropy between the components of the turbulence intensity. As it must be, at the poles the difference between the horizontal components vanishes but the vertical component dominates. For faster rotation there is a clear tendency for a *return-to-isotropy*. A similar crossover behaviour for anisotropic turbulence intensities has also been found for rotating convection-turbulence fields by Chan (2001).

#### 5 DISCUSSION

We have shown that the existence of negative correlations  $Q_{R\phi}$  for turbulent Kepler discs can be explained as being due to a special sort of anisotropy. The presented first-order smoothing turbulence theory only allows negative angular momentum transport,  $\rho Q_{R\phi}$ , in the case where the radial turbulence intensity,  $\langle u_R'^2 \rangle$ , dominates the azimuthal intensity,  $\langle u_\phi'^2 \rangle$ . For anisotropic turbulence, apart from the viscosity term an extra term appears in the relation between the stress and the rate of strain which becomes negative for the radially dominated anisotropy. On the other hand, a positive angular momentum transport requires isotropy or azimuthally dominated anisotropy.

We have checked this finding with a special simulation, where a standard alpha disc becomes convectively unstable. The convective pattern is used as a turbulence model in the sense of the presented consideration. There is no chance, of course, to analyse the background turbulence for anisotropic flow pattern.

The correlation tensor of the convective flow has been established. Its off-diagonal component,  $Q_{R\phi}$ , is indeed negative, where the turbulence is radially dominated (Figs 2–4). The effect is very clear; it does not depend on the value of the viscosity parameter  $\alpha_{\text{SS}}$ .

One can ask whether an example is known which confirms the relation (27) for the opposite type of anisotropy, i.e. for turbulence with dominating azimuthal intensity and, therefore, positive  $Q_{R\phi}$ . This is the case indeed for a Keplerian flow which is modulated with finite disturbances of a given wave number  $k$ . This hydrodynamic flow is unstable if the amplitude of the flow

disturbance exceeds  $1/2kR$  (Dubrulle 1993). The resulting fluctuations are clearly azimuthally dominated and the resulting angular momentum flux is outwards (Drecker & Rüdiger 2001).

As mentioned, our expression (37) not only suggests a clear tendency for the rotating turbulence towards isotropy in the horizontal plane (vertical components excluded, of course) but there is also an indication that even the sign of the anisotropy changes for rapid rotation. Although the effect remains weak, however, its reality must be checked in detail with new simulations, the results of which shall be presented later.

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