MAGNETOHYDRODYNAMICS Vol. 42 (2006), No. 2–3, pp. 3–11

THE MAGNETOROTATIONAL INSTABILITY OF MHD TAYLOR–COUETTE FLOWS

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To realize the magnetorotational instability (MRI) with liquid metals such as gallium in the laboratory, one needs Reynolds numbers exceeding 10^7 (due to their small magnetic Prandtl number Pm), which seems to be out of the current technical possibilities. We have shown that in the presence of combined axial and azimutal magnetic fields ('helical' fields) a new solution beyond the Rayleigh limit appears, which for small magnetic Prandtl numbers does not depend on Pm. The critical Reynolds number is reduced to $O(10^{3...4})$, which does not make any technical difficulty. For liquid gallium between the perfect-conducting cylinders with $R_{\rm out} = 2R_{\rm in} = 8$ cm the necessary axial magnetic field is ~ 100 Gauss and the axial current, which produces the azimutal field, must have ~ 3000 Amp. The critical rotation rate of the inner cylinder is then less than 1 Hz. The gallium Taylor–Couette experiment PROMISE[1] demonstrates the existence of the MRI in the laboratory.

Introduction. The magnetorotational instability (MRI) has turned out to be a universal and efficient instability solving the angular momentum problem in star formation, also explaining the enormous energy output of quasars. It even generates the turbulence of the interstellar medium eventually leading to extended galactic magnetic fields. Today it is a widely accepted phenomenon that the presence of weak magnetic fields makes most of the astrophysically relevant shear flows unstable and turbulent. It is, however, nearly unknown in experimental physics though. The verification in the laboratory will be a major step of understanding the instability and the route to turbulence.

1. Theory. The rotation law in the infinite Taylor–Couette flow is

$$\Omega(R) = a + \frac{b}{R^2}, \tag{1}$$

where a and b are fixed by the angularvelocities $\Omega_{\rm in}$ and $\Omega_{\rm out}$ of the inner and the outer cylinders. Rotation laws (1) with $a \to 0$ are called to be at the Rayleigh limit. With $R_{\rm in}$ and $R_{\rm out}$ as the radii of the cylinders, the parameters $\hat{\mu} = \Omega_{\rm out}/\Omega_{\rm in}$ and $\hat{\eta} = R_{\rm in}/R_{\rm out}$ are defined. According to the Rayleigh criterion, the ideal flow is hydrodynamically stable when $a > 0$ or

$$\hat{\mu} > \hat{\eta}^2. \tag{2}$$

The viscosity stabilizes the flow insofar as it becomes unstable only if the inner cylinder rotates sufficiently fast.

If the fluid is electrically conducting and an axial magnetic field is applied (Fig. 1), then, after the old results, the criticalReynolds number for the inner rotating cylinder grows with the growing magnetic field (Chandrasekhar 1961). Velikhov (1959), however, has discovered a magnetic shear-flow instability, which

[1] "PotsdamROssendorfMagneticInStabilityExperiment"

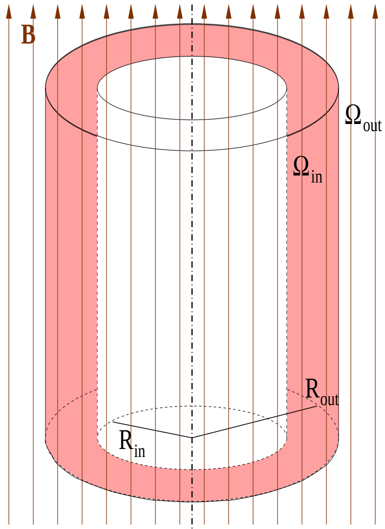

Fig. 1. MHD Taylor-Couette experiment with axial field. $R_{\rm in}$ – radius of the inner cylinder, $R_{\rm out}$ – radius of the outer cylinder, $\Omega_{\rm in}$, $\Omega_{\rm out}$ are the rotation rates of the inner and the outer cylinder.

is now called the 'magnetorotational instability' (MRI). He found that for the ideal hydromagnetic Taylor–Couette flow the Rayleigh criterion for stability changes from (2) to

$$\hat{\mu} > 1. \tag{3}$$

The hydrodynamic Taylor–Couette flow is stable if its angular momentum increases with radius, but the hydromagnetic Taylor-Couette flow is only stable if the angular velocity itself increases with radius. This remains true also for non-ideal fluids. The MRI reduces the critical Reynolds number for weak magnetic field strengths for a hydrodynamically unstable flow and it destabilizes the otherwise hydrodynamically stable flow for $\hat{\eta}^2 < \hat{\mu} < 1$. The MRI exists in hydrodynamically unstable flows ($\hat{\mu} < \hat{\eta}^2$), but only for sufficiently high values of the electric conductivity. We are interested in the stability of the basic state (1) against axisymmetric and nonaxisymmetric perturbations.The solutions of the linearized MHD equations are considered in their modal representation with $\exp({\rm i}(kz+\omega t))$. The dimensionless numbers of the problem are the magnetic Prandtl number Pm, the Hartmann number Ha and the Reynolds number Re,

$${\rm Pm} = \frac{\nu}{\eta}, \qquad {\rm Ha} = \frac{B_0 R_0}{\sqrt{\mu_0 \rho \nu \eta}}, \qquad {\rm Re} = \frac{\Omega_{\rm in} R_0^2}{\nu}, \tag{4}$$

where μ_0 is the permeability, and $R_0 = (R_{\rm in} D)^{1/2}$ with $D = R_{\rm out} - R_{\rm in}$. All other quantities have their usual meaning. Fig. 2 (top) shows the neutral stability, $\Im(\omega) = 0$, for axisymmetric modes for containers with both conducting and insulating walls with resting outer cylinder and for fluids of various magnetic Prandtl number. Re $= 68$ is the classical hydrodynamic solution for the resting outer cylinder and $\hat{\eta} = 0.5$ (see Chandrasekhar 1961). There is a strong difference of the bifurcation lines for Pm $\gtrsim 1$ (high conductivity) and Pm $\ll 1$ (low conductivity). For fluids with low electrical conductivity, the magnetic field only suppresses the instability so that all the critical Reynolds numbers strongly exceed the value 68.

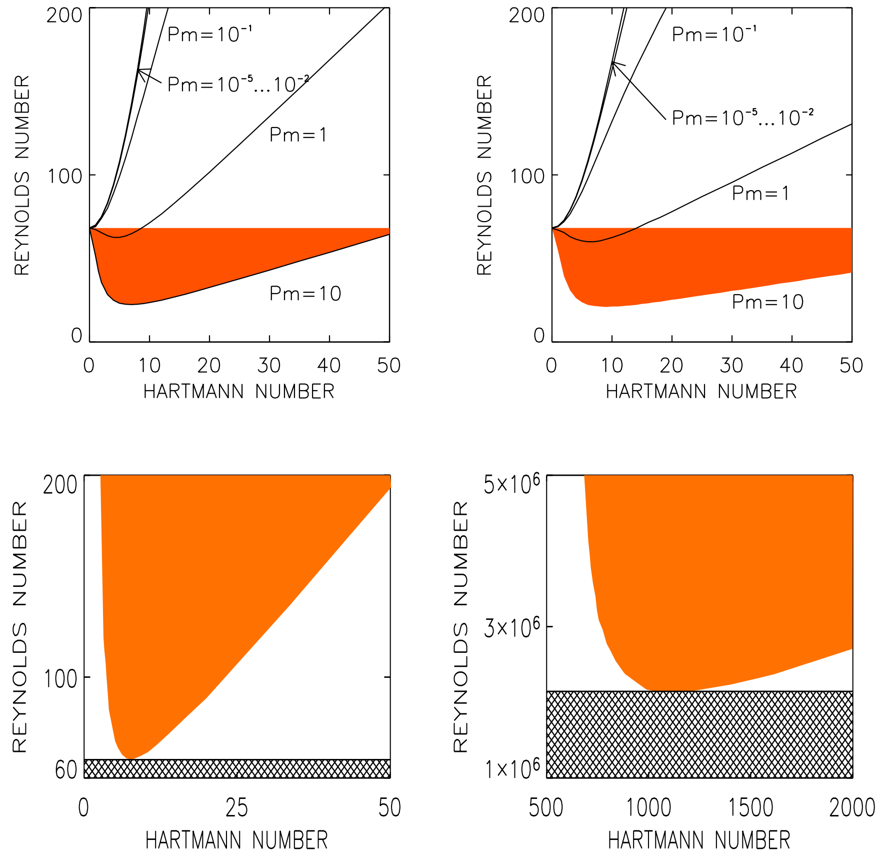

Fig. 2. Bifurcation diagram for axisymmetric modes, $\hat{\eta} = 0.5$. Top: resting outer cylinder ($\hat{\mu} = 0$) of conducting (left) and insulating material (right). Bottom: rotating outer cylinder with $\hat{\mu} = 0.33$ of conducting material for $\mathrm{Pm} = 1$ (left) and $\mathrm{Pm} = 10^{-5}$ (liquid sodium, right). The instability domain is filled, fluids in the cross-hatched area are always stable. From Rüdiger & Shalybkov (2002) and Rüdiger, Schultz & Shalybkov (2003).

For small magnetic Prandtl number the stability lines hardly differ. The opposite is true for $\mathrm{Pm} \gtrsim 1$. Then the resulting critical Reynolds numbers *Re* are smaller than 68. The magnetic fields with small Hartmann numbers support the instability rather than suppress it. This effect becomes more effective for the increasing Pm, but it vanishes for stronger magnetic fields. The MRI only exists for weak magnetic fields and high enough electric conductivity and/or microscopic viscosity.

With a *rotating* outer cylinder, the rotation profile (1) becomes flatter and the hydrodynamic instability disappears when $\hat{\mu} \to \hat{\eta}^2$ (Rayleigh limit, here for $\hat{\mu} \to 0.25$). The magnetic-induced instability, however, is hardly influenced by the new rotation profile (Fig. 2, bottom). There is always a minimum Reynolds number for a certain Hartmann number, where the instability sets in. As we have demonstrated by numerical simulations, it is the magnetic Reynolds number

$$\mathrm{Rm} = \mathrm{Re}\ \mathrm{Pm}, \tag{5}$$

which controls the instability (Fig. 3). In the same way, the Hartmann number

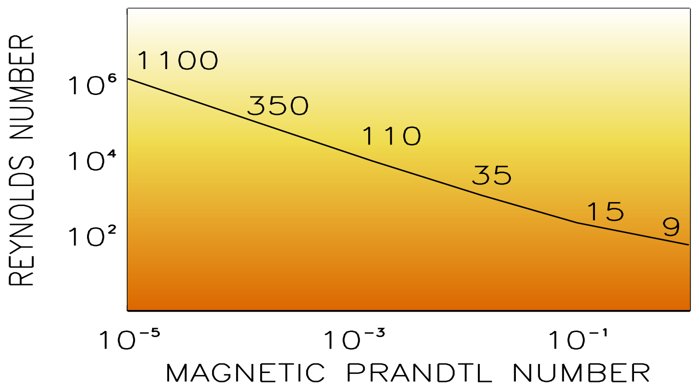

Fig. 3. The critical (minimum) Reynolds numbers vs. magnetic Prandtl numbers marked with those Hartmann numbers, for which the Reynolds number is minimum. $\hat{\eta} = 0.5$, $\hat{\mu} = 0.33$, conducting walls. The resulting magnetic Reynolds number is of order 20. From Rüdiger, Schultz & Shalybkov (2003).

must be replaced by the Lundquist number

$$\mathrm{S} = \mathrm{Ha}\sqrt{\mathrm{Pm}} \tag{6}$$

in order to get results almost independent of Pm. Because of the high value of η for liquid metals (exceeding 10^3 cm^2/s), it is not easy to reach magnetic Reynolds numbers of the required order of $1 \dots 10$. This is the main reason why the MRI has not yet been observed in the laboratory. The needed rotation rates of the inner cylinder are so high that for real containers bounded in z the resulting end-effects are so strong that the experiment is strongly modified (Hollerbach & Fournier 2004). One can even expect, on the other hand, that a flow with such a high Reynolds number becomes non-linearly unstable so that a completely new situation arises.

One could argue that the Taylor–Couette flow in the Rayleigh limit becomes unstable for much lower Reynolds numbers. This is indeed true, as this limit forms an exceptional case. Inspecting the equations, one immediately finds the scalings

$$\mathrm{Re} \propto \mathrm{Pm}^{-1/2}, \qquad\qquad \mathrm{Ha} \propto \mathrm{Pm}^0 \tag{7}$$

so that it is enough to compute the values for Pm = 1 (see Fig. 4). The minimum Reynolds number is 66 for Ha $\simeq$ 7. Hence, Re = 20.870 for sodium and Re = 66.000 for gallium. Such small values seem to be much promising. As we have argued, however, an extreme accuracy in the realization of the Rayleigh-limit condition $\hat{\mu} = \hat{\eta}^2$ would be needed for related experiments. Already very small deviations from this condition lead to completely different values. All in all, there is only little hope to realize the MRI in the Taylor–Couette experiments with liquid metals under exclusive use of axial magnetic fields.

2. Experiments with helical fields. The given solutions of marginal instability of axisymmetric modes are stationary. One could ask for the character of the solutions if an extra toroidal magnetic field is applied (Hollerbach & Rüdiger

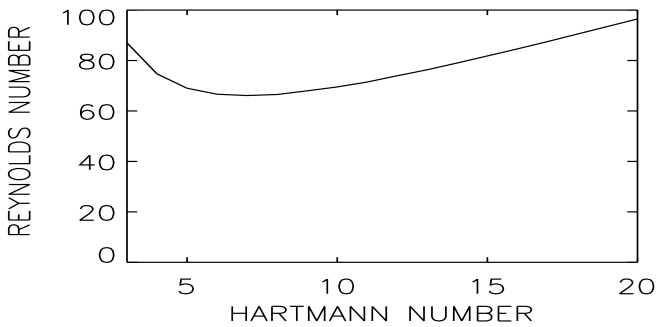

Fig. 4. The critical Reynolds numbers vs. Hartmann numbers *at the Rayleigh limit* for Pm = 1. $\hat{\eta} = 0.5$, $\hat{\mu} = 0.25$. The curve can be transformed by (7) to be valid for smaller magnetic Prandtl numbers.

2005). We know that current-free toroidal fields alone do not support the stability of the Taylor–Couette flow (Michael 1954, Velikhov 1959, Tayler 1973). If axial fields *and* current-free toroidal fields of the same order exist in the container, however, completely new solutions appear, which are oscillating. Stationary modes do not longer exist. The Reynolds number of the flow and the Hartmann number of the axial field, which are necessary to induce the MRI instability, are strongly reduced by the toroidal field.

From the azimutal component of the induction equation one finds the solution

$$B_\phi = \frac{\beta B_0 R_{\rm in}}{R}, \tag{8}$$

which is current-free within the gap.The parameter β denotes the ratio of the toroidal field to the constant axial field B_0. The toroidal field B_ϕ is maintained by an electric current along the central axis with

$$J = 5\beta B_0 R_{\rm in}, \tag{9}$$

where J is measured in Amp, B_0 in Gauss and $R_{\rm in}$ in cm. In order to understand the results, it is useful to consider the induction equation for the fluctuating toroidal field component, i.e.,

$$\frac{\partial B'_\phi}{\partial t} - \eta \Delta B'_\phi = {\rm rot}_\phi \big(\bar{\boldsymbol{u}}_\phi \times \boldsymbol{B}'_{\rm pol} + \boldsymbol{u}'_\phi \times \bar{\boldsymbol{B}}_{\rm pol} + \boldsymbol{u}'_{\rm pol} \times \bar{\boldsymbol{B}}_\phi \big), \tag{10}$$

where $\boldsymbol{u}_{\rm pol}$ and $\boldsymbol{B}_{\rm pol}$ denote the poloidal components of $\boldsymbol{u}$ and $\boldsymbol{B}$. Chandrasekhar (1961) canceled the last two terms on the right of this equation and did not obtain the MRI. It occurs if only the last term in (10) is canceled. If the last term in (10) is retained, then the resulting Reynolds number loses its ${\rm Re} \propto {\rm Pm}^{-1}$ dependence and is reduced by 3 orders of magnitude. The toroidal field modifies the extremely high gradient of the line of neutral stability obtained for $\beta = 0$ and $\hat{\mu} \gtrsim 0.25$. Its enormous steepness is reduced and it starts at much lower Reynolds numbers. For $\beta \gtrsim 2$, we find critical Reynolds numbers of order 10^3 (Fig. 5, left). This is a dramatic reduction of the values of order 10^6, which are characteristic for $\beta = 0$. The toroidal magnetic field (which alone is stable) strongly supports the MRI as a second-order effect in B_ϕ. Simultaneously, the solutions become traveling waves with a phase velocity depending of the sign of B_ϕ.

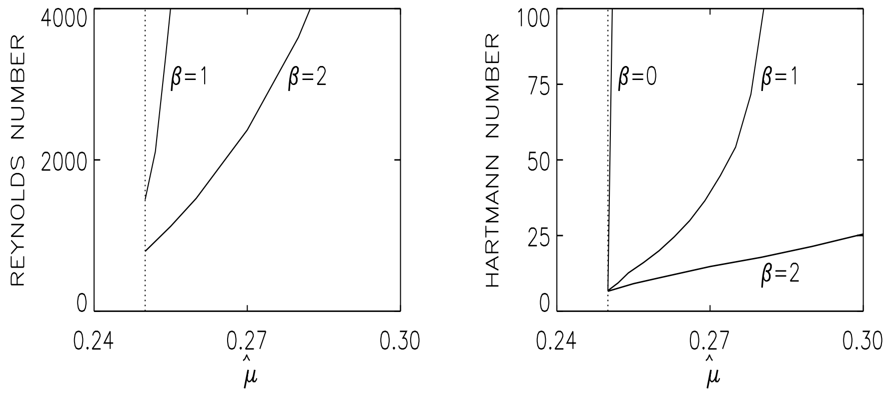

Fig. 5. Left: The critical Reynolds numbers beyond the Rayleigh line (here $\hat{\mu} = 0.25$), for axisymmetric modes with toroidal fields of the order of the axial field. Notice that the presence of the toroidal field strongly supports the magnetorotational instability of the differential rotation. Right: The Hartmann numbers of the solutions. $\mathrm{Pm} = 10^{-5}$ (sodium), the walls are perfect conductors.

The corresponding Hartmann numbers for these instabilities are also strongly reduced by the inclusion of a toroidal field (Fig. 5, right).Table 1 shows results for $\hat{\mu} = 0.27$ (i.e., slightly beyond the Rayleigh limit) and $\mathrm{Pm} = 10^{-6}$ (gallium). The first three columns give the resulting non-dimensional quantities. The last four columns give the dimensional quantities for $R_{\mathrm{in}} = 4$ cm and $R_{\mathrm{out}} = 8$ cm. A container with the aspect ratio $\Gamma = H/D = 10$ (H as the height of the container)

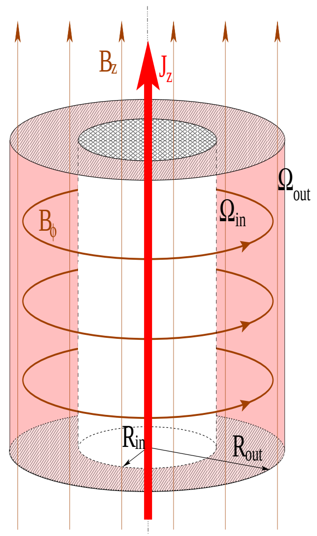

Fig. 6. Prototype of PROMISE with $R_{\mathrm{in}} = 4$ cm, $R_{\mathrm{out}} = 8$ cm, height = 40 cm ($\Gamma = 10$). Cylinders from copper, between the cylinders liquid gallium, end plates rotating along with the outer/inner cylinder.

Table 1. Characteristic values for gallium experiments ($\mathrm{Pm} = 10^{-6}$) with conducting cylinders with $\hat{\eta} = 0.5$ and $\hat{\mu} = 0.27$. $R_{\rm in} = 4$ cm and $R_{\rm out} = 8$ cm. B_ϕ denotes the toroidal field at the inner cylinder, the magnetic fields are given in Gauss, the current J in kAmp (Rüdiger et al. 2005).

β	Re	Ha	k	$f_{\rm in}$ [Hz]	B_0 [G]	B_ϕ [G]	J [kAmp]
0	1.02×10^7	1700	1.8	318	9460	0	0
[0.5ex]2	2386	14.6	1.3	0.08	80	160	3.20
[0.5ex]4	846	9.5	1.9	0.03	52	208	4.16

and with resting and/or rotating end plates forms the prototype of the MHD gallium experiment PROMISE, which is planned by the Astrophysical Institute of Potsdam and the Forschungszentrum Rossendorf to realize the MRI in the laboratory (see Fig. 6).

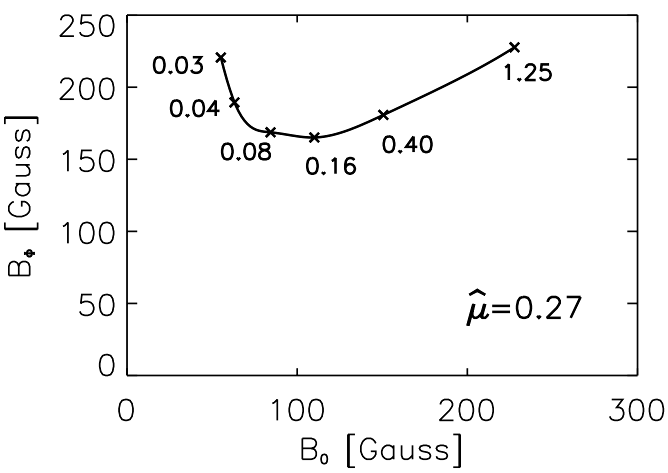

Fig. 7. The line of marginal stability for the given field components. T he line is marked with the critical values for the minimum rotation rate of the inner cylinder in Hz. The cylinders are perfect conductors; the same experiment as in Table 1.

For $\beta \gtrsim 2$ we find the critical Reynolds numbers independent of the magnetic Prandtl number [2]. This is the same result noted by Hollerbach & Rüdiger (2005) for insulating boundaries. More quantitatively, we note that for $\beta \gtrsim 2$ the required rotation rates of the inner cylinder are reduced to less than1 Hz, the axial fields to about 100 Gauss, and the axial currents to$O(1)$ kAmp. Fig. 7 combines the numerical results by elimination of the factor β. A minimum exists for the axial current at axial fields of about 140 Gauss.

3. Experiments for smaller gaps. The question arises whether or not containers with smaller gaps are better to use in the laboratory. Of course, TC experiments with small distances D between the cylinders were cheaper and they can reach higher aspect ratios that is important for the undisturbed realization

[2]only for $\mathrm{Pm} < 1$, for $\mathrm{Pm} \simeq 1$ the influence of the toroidal field vanishes

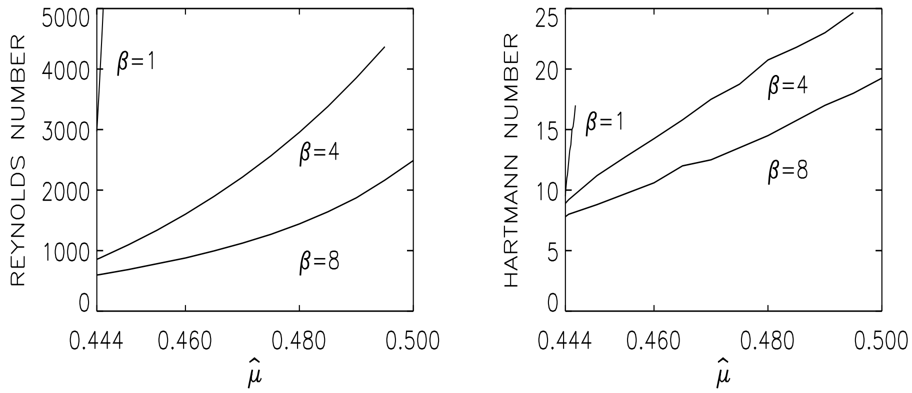

Fig. 8. A small-gap container with $\hat{\eta} = 0.67$. Left: The critical Reynolds numbers for $\mathrm{Pm} = 10^{-6}$ beyond the Rayleigh line (here $\hat{\mu} = 0.444$). Right: The related critical Hartmann numbers.

of Taylor vortices. In hydrodynamic experiments, the vertical size of one Taylor vortex is about D so that Γ approximates the number N of the vortices along the rotation axis. In other words, one observes a pair of Taylor vortices only in experiments with $\Gamma \geq 2$. For hydromagnetic Taylor–Couette flows, the aspect ratio must strongly exceed the value of 2 in order to realize a pair of Taylor vortices in the experiment. The vertical size of a single Taylor vortex in units of the gap width is

$$\frac{\delta z}{D} = \frac{\pi}{k}\sqrt{\frac{R_{\mathrm{in}}}{D}}. \tag{11}$$

For the small gap with $\hat{\eta} = 0.8$ one finds $\delta z/D = 2\pi/k$. According to our results, it is always $k \lesssim 5$ for $\beta \lesssim 8$ so that always $\delta z > D$, the cells are thus prolate. The number of cells proves to be

$$N = \frac{H}{\delta z} = \frac{k}{2\pi}\frac{H}{D} \tag{12}$$

which is *smaller* than the aspect ratio H/D. In order to get at least two vortices, the aspect ratio must exceed the value $\simeq 4$.

Exceeding the aspect ratios of order 5 for a small gap does not make any technical problem. The basic problem for experiments with such containers, however, is the large amplitude of the necessary axial electric current (see Table 2).

Table 2. A small-gap container with $R_{\mathrm{in}} = 4$ cm, $R_{\mathrm{out}} = 6$ cm and for$\hat{\mu} = 0.46$ and $\mathrm{Pm} = 10^{-6}$ (gallium). $\Gamma^{(2)}$ is the minimum aspect ratio to allow $(N=)2$ single vortices.

β	Re	Ha	k	f_{in} [Hz]	B_0 [G]	B_ϕ [G]	J [kAmp]	$\Gamma^{(2)}$
2	7200	24.7	1.46	0.45	192	383	7.66	8.6
4	1602	14.3	2.32	0.10	111	444	8.88	5.4
8	877	10.6	3.24	0.06	82.4	659	13.2	3.9

4. Conclusion. We have demonstrated the feasibility of TC-experiments with liquid metals to realize the MRI in the laboratory. While the necessary Hartmann numbers do not exceed the experimental possibilities, the resulting Reynolds numbers 10^6 (for sodium) and 10^7 (for gallium) lead into the non-linear domain. Both critical numbers can basically be reduced by working in the Rayleigh limit ($\hat{\mu} = \hat{\eta}^2$), but then even the smallest deviation from this condition would destroy all excitation. By inclusion of a toroidal field component (due to an axial current), the necessary Reynolds/Hartmann numbers of experiments (slightly) beyond the Rayleigh limit are strongly reduced. So far, the effect has been demonstrated only for infinite cylinders. It is, however, necessary also to develop codes, by which the influence of the end plates can be investigated. This the more as for the small Hartmann numbers under consideration the Hartmann layers inside the end plates become rather thick.

5. Acknowledgement. An anonymous referee is acknowledged for his interesting remarks.

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Received 06.07.2006