An Integral Equation Approach to Kinematic Dynamo Models

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Abstract

The paper deals with dynamo models in which the induction effects act within a bounded region surrounded by an electrically conducting medium at rest. Instead of the induction equation, an equivalent integral equation is considered, which again poses an eigenvalue problem. The eigenfunctions and eigenvalues represent the magnetic field modes and corresponding dynamo numbers.

In the simplest case, that is for homogeneous conductivity and steady fields, this integral equation follows immediately from the Biot-Savart law. For this case, numerical results are presented for some spherical and elliptical axisymmetric $\alpha^2\omega$-dynamo models. For a large class of models the interesting feature of dipole-quadrupole degeneration is found.

Using Green's function of a Helmholtz-type equation, we derive a more general integral equation, which applies to time-dependent magnetic field modes, too, and gives us some insight into the spectral properties of the integral operators involved. In particular, for homogeneous conductivity the operator is compact and thus bounded, which leads to a necessary condition for dynamo action.

KEY WORDS: $\alpha^2\omega$-dynamos, kinematic dynamos, mean-field electrodynamics
1 Introduction

Most cosmic magnetic fields (like those of planets, the Sun and galaxies) are believed to be generated and maintained against dissipation by dynamo mechanisms. In many astrophysical objects, turbulent motions play a crucial rôle and consequently the electromagnetic fields show turbulent features, too. We will therefore adopt the mean-field concept (see e. g. Krause & Rädler, 1980) that proved to be useful in this context. The original equations of magnetohydrodynamics can be easily recovered as a special case, putting to zero the coefficients describing turbulence effects.

The mean-field dynamo problem as we will discuss it here is defined by Maxwell's equations (in MHD approximation)

\[
\begin{align*}
\text{curl} \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \quad (1. \ a) \\
\text{div} \mathbf{B} &= 0 \quad (1. \ b) \\
\text{curl} \mathbf{B} &= \mu_0 \mathbf{j} \quad (1. \ c)
\end{align*}
\]

and Ohm's law

\[\mathbf{j} = \sigma (\mathbf{E} + \mathcal{F}).\] (2)

The induced electromotive force \(\mathcal{F}\) is due to the mean and turbulent motions. We will use the mean-field relation

\[\mathcal{F} = \mathbf{u} \times \mathbf{B} + \alpha \mathbf{B} - \beta \text{curl} \mathbf{B},\] (3)

but in principle all the equations given below that involve \(\mathcal{F}\) are applicable also to the general nonlinear case. \(\mathbf{B}, \mathbf{E}, \mathbf{j}\) and \(\mathbf{u}\) are magnetic flux density, electric field strength, electric current density and velocity field of the fluid, all understood as mean quantities. We assume the fluid to have the magnetic permeability \(\mu_0\) of vacuum; \(\sigma\) denotes the electrical conductivity. The turbulent motions are assumed to give rise to an \(\alpha\)-effect, that is, an electromotive force showing a component parallel to \(\mathbf{B}\). Here \(\alpha\) is considered to be a symmetric tensor; any antisymmetric part can be taken into account by a proper modification of \(\mathbf{u}\). The parameter \(\beta\) characterises turbulent dissipation of the magnetic field and is, for sake of simplicity, supposed to be scalar. We suppose that the induction effects represented by \(\mathbf{u}, \alpha\) and \(\beta\) act only within a bounded region \(\mathcal{D}\), which we will call dynamo region. Both \(\mathcal{D}\) and all infinite space surrounding it are supposed to be electrically conducting, i.e. \(\sigma > 0\) everywhere.

The fields \(\mathbf{B}, \mathbf{E}\) and \(\mathbf{j}\) are supposed to be square integrable. Note that the square integrability of \(\mathbf{B}\), i.e. \(\int B^2\, dx^3 < \infty\), implies the absence of currents at infinity. In connection with the equation \(\Delta \mathbf{B} = 0\) holding outside the bounded dynamo region for steady fields and homogeneous conductivity, it also excludes magnetic flux from infinity. This can be seen from the asymptotics (15) in Section 2.2, that has the property \(B_r \to 0\) for \(r = |x| \to \infty\). These two physical requirements replace the assumption that \(\mathbf{B}\) decays like a dipole field, which holds only in the case of a dynamo surrounded by an insulator.
In this paper, we are interested mostly in the kinematic dynamo problem, where the coefficients \( u \) and \( \alpha \) in Equation (3) are given functions of position and the resulting problem is linear in \( B \). Actually, most of our discussion remains valid in the more general case \( u = u(x; B) \), \( \alpha = \alpha(x; B) \), \( \beta = \beta(x; B) \), and we will mention the relation with this case where it seems necessary.

The traditional approach to the dynamo problem is based on the induction equation, which can be easily derived from Equations (1)-(3). For the case of homogeneous conductivity, \( \sigma = \text{const} \) (everywhere), it reads

\[
\frac{\partial B}{\partial t} - \Delta B = C \text{curl}(u \times B + \alpha B - \beta \text{curl}B) , \quad \text{div}B = 0 .
\]

As usual, we have introduced here dimensionless variables based on a unit length \( L \) (typically the extension of the dynamo region \( \mathcal{D} \)) and a time \( \mu_0 \sigma L^2 \) (the diffusion time). \( u \) and \( \alpha \) are measured in a unit velocity \( U \) (typically the maximum value of \( |u| \) or \( || \alpha || \)) and \( \beta \) in \( U \cdot L \). We have not merged the term \( \text{curl} \beta \text{curl}B \) with the Laplacian on the left hand side; this is useful for our later discussion of the case of non-homogeneous conductivity. Finally, we have introduced the dynamo number

\[
C = \mu_0 \sigma U L ,
\]

which is a dimensionless measure of the strength of the induction effects compared to Ohmic dissipation.

On this level, the dynamo problem consists in finding solutions \( B \) of (4) which do not decay as \( t \to \infty \) in the sense that \( \| B \|_2 := \int B^2 \, dx^3 > K \quad \forall t \) with a fixed lower bound \( K > 0 \). In particular, we will look for solutions of time-dependence \( \sim \exp(\gamma t) \). Then, the induction equation (4) poses an eigenvalue problem for the complex growth rate \( \gamma = \gamma_r + i \gamma_i \), or, if we are interested in solutions with given \( \gamma_r \), a two-parameter eigenvalue problem for \( C \) and \( \gamma_i \). We call critical dynamo numbers the values of \( C \), for which there exist such exponentially "growing" solutions with \( \Re \gamma = 0 \).

Several methods have been used to solve the dynamo problem at different levels of idealisation, the most common approaches being spectral methods, finite-difference methods, and asymptotic analysis. The majority of the published models supposes vacuum outside the dynamo region, and their results cannot be directly compared with ours.

Finite-difference methods have to cope with non-local boundary conditions at the outer boundary of the finite volume of calculation (cf. Bräuer & Rädler, 1986, for the case of vacuum surrounding the dynamo). In order to overcome this difficulty, one usually inflates the domain of calculation considerably, which increases numerical cost. When a Cauchy problem is solved instead of an eigenvalue problem, only the leading mode (or a few leading ones) can be calculated and no understanding of the overall spectral structure is obtained. There are however also finite-difference models that solve for the whole spectrum of the kinematic dynamo problem.
In this paper, we propose another approach to solve the dynamo problem. Instead of reducing Equations (1), (2) to a eigenvalue problem for a differential operator we transform it into an eigenvalue problem for an integral operator, which we then solve numerically. The simplest version of this integral-eigenvalue equation for $B(x)$ has the dimensionless form (cf. Section 2.1)

$$\int_{\mathcal{D}} \frac{(x-x') \times [u(x') \times B(x') + \alpha(x') B(x') - \beta(x') \text{curl} B(x')]}{|x-x'|^3} \, dx'^3 = -\frac{4\pi}{C} B(x) \quad (6)$$

and holds in the case of homogeneous conductivity and steady electromagnetic fields. Equation (6) is the continuous analog of a matrix eigenvalue problem. Solving it on the bounded region $\mathcal{D}$ has several advantages compared to the induction equation (4). First, no boundary conditions have to be specified, since the boundary $\partial \mathcal{D}$ is a set of measure zero and does not contribute to the integral. Moreover, the action of the whole current system in the conducting medium outside the dynamo region is contained in (6) in a consistent way, but its explicit treatment is unnecessary: we can solve for the field $B(x)$ in $\mathcal{D}$ alone.

Our assumption that $\sigma$ be constant in the whole space surrounding the dynamo region is somewhat opposite to the (mostly vacuum surrounded) traditional dynamo models. For stellar and galactic dynamos, however, the surrounding space is also a conductor, and our assumption seems more realistic than assuming vacuum. Moreover, the description of vacuum poses some conceptual problems in magnetohydrodynamics because quasi-stationarity is well fulfilled by all cosmic plasmas, but not by vacuum (cf. Sokoloff, 1997). In a quasi-stationary system, electromagnetic fields propagate at a time scale much shorter than both material advection time $\tau_{\text{ad}} = L/U$ and diffusion time, $\tau_{\text{diff}} = \mu_0 \sigma L^2$. But for $\sigma \to 0$, the latter becomes infinitely small.

In this light, it is necessary to examine dynamo models embedded in conducting space in order to discuss some fundamental aspects of cosmic dynamo theory.

In Section 2 we derive the integral equation for the case of steady fields and obtain the far-field asymptotics for steady dynamos surrounded by a conducting medium. In Section 3 we restrict ourselves to homogeneous conductivity and present numerical results for some $\alpha^2 \omega$-dynamo models. Section 4 is devoted to the mathematical foundation of the integral equation and its generalisation to the time-dependent case. A necessary condition for self-excited dynamos with homogeneous conductivity is established and a generalised integral equation for varying conductivity is given.
2 The Biot-Savart law and the integral eigenvalue equation for $B$

2.1 The integral equation

For a given square integrable current field $j$, Equations (1.b) and (1.c) determine $B$ in the following way (cf. for example Jackson, 1980):

$$
B(x) = -\frac{\mu_0}{4\pi} \int \frac{(x-x')\times j(x')}{|x-x'|^3} \, dx'
$$

(7)

$$
= \frac{\mu_0}{4\pi} \int \frac{\text{curl}' j(x')}{|x-x'|} \, dx'.
$$

(8)

Equation (7) is the well-known Biot-Savart law; (8) is obtained by means of integration by parts\(^1\) and shows that an arbitrary gradient can be added to or subtracted from $j(x)$ without changing any of the two integrals (7) or (8). Note that, while $j$ is solenoidal in accordance with Maxwell’s equations, $\text{div} \, j = 0$, the vector field $j_1 = j - \text{grad} \psi$ is not solenoidal (otherwise it is either trivial, or does not vanish for $|x| \to \infty$) and cannot be interpreted as a physical current. Nevertheless we can apply the integral (7) to the pseudo-current field $\tilde{j}$ in order to get the magnetic field caused by the physical current density $j$. This strategy is illustrated in Figure 1 for the case of the first mode of the dynamo model of Krause & Steenbeck (1967); cf. Section 3, Equation (27) for the specification of this model.

Returning to the dynamo problem (1)–(2), we restrict ourselves to steady fields and suppose that the electric conductivity is constant, $\sigma = \sigma_{\text{hom}} > 0$ everywhere (see the discussion below). We insert Ohm’s law (2), into the integral (7) and, exploiting the fact that $\sigma E = -\sigma \text{grad} \Phi$ is a gradient (for this step, the homogeneous conductivity $\sigma$ is crucial), we can omit the term containing the electric field $E$. In dimensionless variables as mentioned in the Introduction, we thus obtain

$$
\left(\frac{1}{\mu_0} \mathcal{B}\right)(x) := \int \frac{(x-x')\times \mathcal{F}(x')}{|x-x'|^3} \, dx' = -\frac{4\pi}{C} B(x)
$$

(9)

with $\mathcal{F}$ given by Equation (3).

Equation (9) was first given by Roberts (1967, 1994) for laminar dynamos with homogeneous conductivity ($\alpha = 0$, $\beta = 0$). It is an integral-eigenvalue equation in $B(x)$; the spectrum of eigenvalues $\{-4\pi/C\}$ gives the critical dynamo numbers $C$, and the eigenfunctions $B(x)$ represent the corresponding eigenmodes of the magnetic field.

In the nonlinear case, when $u$, $\alpha$ or $\beta$ depend on $B$, Equation (9) is a nonlinear integral-eigenvalue problem, which introduces some complications (but not principal ones) into the mathematical and numerical treatment.

\(^1\)Thereby, in addition to (8), a surface integral $\int \frac{1}{|x-x'|} \, dx'$ occurs. But as the domain of integration in (7) and (8) is the whole space, $\mathcal{V} = \mathbb{R}^3$, this integral vanishes for any square integrable current field.
It is very important that the integration in Equation (9) is actually not over the whole space, but only over the compact dynamo region
\[ D = \{ x \mid \sigma(x) \neq 0 \lor u(x) \neq 0 \lor \beta(x) \neq 0 \} . \]  
This relieves us of deep mathematical and numerical problems connected with unbounded integral operators.

Our assumption that \( \sigma \equiv \text{const} \) can easily be dropped, since any variation in the conductivity \( \sigma \) can be formally ascribed to the diffusivity \( \beta \). Let us suppose that \( \sigma = \sigma_{\text{ext}} \) outside \( \mathcal{D} \) and that electrical conductivity and turbulent diffusivity within the dynamo region are given by \( \tilde{\sigma} \) and \( \tilde{\beta} \), respectively. Then the simple transformation
\[ \sigma := \sigma_{\text{ext}} \]  
\[ \beta(x) := \tilde{\beta}(x) + \frac{1}{\mu_0} \left( \frac{1}{\tilde{\sigma}(x)} - \frac{1}{\sigma_{\text{ext}}} \right) . \]
gives us again a formally homogeneous conductivity \( \sigma \) and a diffusivity \( \beta(x) \) with finite support. Hence, Equation (9) is applicable whenever \( \sigma \) is constant (but \( \neq 0 \)) outside some bounded region.

In the case \( \beta \equiv 0 \), the integral equation takes the form
\[ \int \frac{(x-x') \times [u(x') \times B(x') + \tilde{\sigma}(x') B(x')] \, dx'}{|x-x'|^3} \, dx = -\frac{4\pi}{C} B(x) . \]
and the operator $\hat{A}$ from (9) is an integral operator with weak singularity at $x' = x$ (we have $(x-x')/|x-x'|^3 = O(1/|x-x'|^3)$, so that the integral (9) exists in the strict sense). Therefore, and because the domain of integration is bounded, $\hat{A}$ is bounded and, moreover, compact (cf. Kress, 1989, theorem 2.21). Hence, Riesz' first theorem (theorem 3.1 or particularly 3.11 in the book of Kress) tells us immediately that the spectrum of $\hat{A}$ is countable (i.e. discrete) and has no other point of accumulation than 0 (corresponding to $C = \infty$). As the integral operator $\hat{A}$ is not self-adjoint, there is no guarantee that the spectrum contains values other than 0. We are of course interested only in the cases in which eigenvalues other than zero exist and will discuss only them.

2.2 The far field

One result that can be easily derived from the integral equation (9) is the asymptotic behaviour of the steady field $B$ far from the dynamo region:

$$B(x) = -\frac{\mu_0\sigma}{4\pi} \frac{x}{|x|^3} \times \int \mathcal{F}(x') \, dx' + O\left(\frac{1}{r^3}\right)$$

$$= \frac{\mu_0}{4\pi} L[I_1] \frac{\sin \theta}{r^2} e_\phi + O\left(\frac{1}{r^3}\right) \quad \text{for } r = |x| \to \infty$$

(in dimensional form), where $(r, \theta, \phi)$ represent polar coordinates whose $z$-axis is chosen parallel to the vector $LI_1 := \sigma \int \mathcal{F}(x') \, dx'$. This formula is valid not only in the linear case, but rather for an arbitrary induced electromotive force $\mathcal{F}$ of finite support. Moreover, it applies even to dynamos with varying conductivity $\sigma(x)$, supposed that $\lim_{|x| \to \infty} \sigma(x) = \sigma_\infty > 0$ exists. In Section 4.2 we will show that for time-dependent modes the fields decay exponentially with $r$, i.e. much faster.

When the dynamo region is embedded into vacuum, the external field allows for multipole expansion and the far-field condition reads that $B(x)$ vanishes far from the dynamo region like a dipole field ($\sim 1/r^3$); this result could also be obtained from Equation (15) in the limit $\sigma \to 0$, $\beta \sigma$ fixed. In the case of a conducting cosmos however, outside the dynamo region there is still a current, driven by the electric field $E$ whose leading term is an electric dipole field: $j \sim E_{\text{dipole}} \sim 1/r^3$. This current leads to a magnetic field $B \sim 1/r^2$, but as this component of $B$ is merely azimuthal, it does not contradict the condition that current and magnetic flux must be localised. Roberts (1967, 1994) mentioned this far-field behaviour already in 1967 and it is not difficult to derive the asymptotics (15) from the discussion of the far-field by Meinel (1989) or, for a concrete example, from the results of Krause & Steenbeck (1967).

For most cosmic dynamos, $\alpha(x)$ is antisymmetric and $u(x)$ and $\beta(x)$ are symmetric with respect to the equatorial plane (cf. Dobler & Rädler, 1997, for a detailed discussion). Then, the field modes appear as symmetric fields $B^S$ or antisymmetric fields $B^A$. Obviously, $\mathcal{F}$ is antisymmetric for symmetric (quadrupole-like), and symmetric for antisymmetric (dipole-like) fields $B(x)$, and the integral $LI_1$ vanishes for
the antisymmetric mode $\mathbf{B}^A$. This implies $|\mathbf{B}^A(x)| = O(1/r^3)$ and leads to the ironic result that for this very common kind of symmetry the first mode with quadrupole symmetry generally decays slower ($|\mathbf{B}^Q(x)| \sim 1/r^2$) than the first dipole-like mode ($\sim 1/r^3$) for $r \to \infty$. In this light, the use of the terms “quadrupole-like/dipole-like” for antisymmetric and symmetric fields seems quite questionable when the fields do not allow for multipole expansion.

### 2.3 The integral equation in cylindrical coordinates

In order to apply the integral equation (9) to concrete dynamo models, we adopt cylindrical coordinates $(\rho, \phi, z)$ and denote the corresponding unit vectors by $\mathbf{e}_\rho$, $\mathbf{e}_\phi$, $\mathbf{e}_z$. It would not be difficult to rewrite the following relations in, say, spherical coordinates. We again suppose homogeneous conductivity, $\sigma = \text{const}$, and $\beta = 0$, since the term $-\beta \mathbf{curl} \mathbf{B}$ in (9) would pose additional difficulties. For the sake of simplicity, we assume the $\alpha$-effect to be isotropic, $\alpha_{ik}(x) = \alpha(x) \delta_{ik}$ with a scalar function $\alpha(x)$. Furthermore, let $\mathbf{u}$ and $\alpha$ be axisymmetric, that is, $u_\rho$, $u_\phi$, $u_z$, and $\alpha$ are independent of $\phi$. Then, the field modes with different azimuthal wave number $m$ are not coupled and we can restrict ourselves to fields $\mathbf{B}(x) = B_\rho \mathbf{e}_\rho + B_\phi \mathbf{e}_\phi + B_z \mathbf{e}_z$ with $B_\rho/\rho(z) = B_\phi/\phi(z) \exp(im\phi)$ for a fixed integer $m$.

Inserting this into the integral-eigenvalue equation (13) and carrying out the integration over azimuth $\phi$, we get after some algebra

$$\hat{\mathbf{I}} \mathbf{B} = -\frac{4\pi}{C} \mathbf{B}$$

(16)

for the column vector

$$\bar{\mathbf{B}} = \begin{pmatrix} \hat{B}_\rho \\ \hat{B}_\phi \\ \hat{B}_z \end{pmatrix},$$

(17)

with the integral operator

$$\hat{\mathbf{I}} = \hat{\mathbf{I}}_u + \hat{\mathbf{I}}_c + \hat{\mathbf{I}}_s + \hat{\mathbf{I}}_{\alpha}$$

(18)

$$\hat{\mathbf{I}}_u \bar{\mathbf{B}} = \int \! dz' \! d\phi' \! du(x,\phi') \left( \begin{array}{ccc} -\rho E_{\rho}^m \hat{B}_\rho & + (z-z') \rho E_{\rho}^m \hat{B}_z \\ + (z-z') \rho E_{\phi}^m \hat{B}_\rho & - \rho E_{\phi}^m \hat{B}_\phi \\ - \rho E_{\phi}^m \hat{B}_{\phi} & + (z-z') \rho E_{\phi}^m \hat{B}_z \end{array} \right)$$

(19. a)

$$\hat{\mathbf{I}}_c \bar{\mathbf{B}} = \int \! dz' \! d\phi' \! du(x,\phi') \left( \begin{array}{ccc} \rho E_{\rho}^m \hat{B}_\rho & - (z-z') \rho E_{\rho}^m \hat{B}_\phi \\ + (z-z') \rho E_{\phi}^m \hat{B}_\rho & - \rho E_{\phi}^m \hat{B}_\phi \\ - \rho E_{\phi}^m \hat{B}_{\phi} & + (z-z') \rho E_{\phi}^m \hat{B}_z \end{array} \right)$$

(19. b)

$$\hat{\mathbf{I}}_s \bar{\mathbf{B}} = \int \! dz' \! d\phi' \! du(x,\phi') \left( \begin{array}{ccc} - (z-z') \rho E_{\rho}^m \hat{B}_\phi & + (z-z') \rho E_{\phi}^m \hat{B}_\rho \\ + (z-z') \rho E_{\phi}^m \hat{B}_\phi & - \rho E_{\phi}^m \hat{B}_\phi \\ - \rho E_{\phi}^m \hat{B}_{\phi} & + (z-z') \rho E_{\phi}^m \hat{B}_z \end{array} \right)$$

(19. c)

$$\hat{\mathbf{I}}_{\alpha} \bar{\mathbf{B}} = \int \! dz' \! d\phi' \! du(x,\phi') \left( \begin{array}{ccc} - (z-z') \rho E_{\rho}^m \hat{B}_\phi & + (z-z') \rho E_{\phi}^m \hat{B}_\rho \\ + (z-z') \rho E_{\phi}^m \hat{B}_\phi & - \rho E_{\phi}^m \hat{B}_\phi \\ - \rho E_{\phi}^m \hat{B}_{\phi} & + (z-z') \rho E_{\phi}^m \hat{B}_z \end{array} \right)$$

(19. d)
Here $\bar{B}_{\theta}/\varphi/z$ stands for $\bar{B}_{\theta}/\varphi/z(\vartheta', \varphi')$ and $E_{1/c,s}^{m}(\vartheta, \varphi', z-z')$. The functions $E_{1/c,s}^{m}$ are integrals over $\varphi'$ and can be expressed in terms of hypergeometric functions. For details see Appendix A.1.

For an axisymmetric $\alpha^2 \omega$-dynamo (i.e. $m=0, \nu = \omega_0 \eta_0$) Equation (16) takes the form

$$- \frac{4\pi}{C} B_{\theta} = -\hat{A} B_{\varphi} \quad (20. \text{a})$$

$$- \frac{4\pi}{C} B_{\varphi} = (\hat{A} + \hat{F}) B_{\theta} + (\hat{D} + \hat{G}) B_{z} \quad (20. \text{b})$$

$$- \frac{4\pi}{C} B_{z} = \hat{E} B_{\varphi} \quad (20. \text{c})$$

with the integral operators

$$(\hat{A}\psi)(\vartheta, z) = \int d\vartheta' d\varphi' \varphi'(\vartheta', \varphi', \varphi'') E_{0}^{0} \psi(\vartheta', \varphi')$$

$$(\hat{D}\psi)(\vartheta, z) = -\int d\vartheta' d\varphi' \varphi'(\vartheta', \varphi', \varphi'') (\nu E_{1/1}^{0} - \varphi E_{1}^{0} \psi(\vartheta', \varphi'))$$

$$(\hat{E}\psi)(\vartheta, z) = \int d\vartheta' d\varphi' \varphi'(\vartheta', \varphi', \varphi'') (\nu E_{c}^{0} - \varphi E_{c}^{0} \psi(\vartheta', \varphi'))$$

$$(\hat{F}\psi)(\vartheta, z) = \int d\vartheta' d\varphi' \varphi'(\vartheta', \varphi', \varphi'') \omega(\vartheta', \varphi', \varphi'') E_{0}^{0} \psi(\vartheta', \varphi')$$

$$(\hat{G}\psi)(\vartheta, z) = \int d\vartheta' d\varphi' \varphi'(\vartheta', \varphi', \varphi'') \omega(\vartheta', \varphi', \varphi'') (\varphi E_{0}^{0} - \varphi E_{1}^{0} \psi(\vartheta', \varphi'))$$

(again the arguments of $E_{0/1}^{0}$ have been omitted for brevity). Equations (20) reflect the fact that differential rotation (the operators $\hat{F}, \hat{G}$) generates only toroidal field from poloidal one, whereas the $\alpha$-effect (operators $\hat{A}, \hat{D}, \hat{E}$) can moreover generate poloidal field from toroidal one.

From the system (20) we can eliminate $B_{\theta}$ and $B_{z}$ and get an integral equation in $B_{\varphi}$ alone,

$$\left[(\hat{D} + \hat{G}) \hat{E} - (\hat{A} + \hat{F}) \hat{A}\right] B_{\varphi} = \left(\frac{4\pi}{C}\right)^{2} B_{\varphi} \quad (26)$$

This kind of reduction is possible only for $m=0$. Equation (26) is the integral equation that we solved numerically in order to get the results shown in Section 3.

### 3 Numerical results

In this section, we present some results obtained by discretising Equation (26), which holds for steady, axisymmetric modes of $\alpha^2 \omega$-dynamos in the case of homogeneous conductivity. The resulting matrix-eigenvalue problem has been solved numerically by standard techniques.
The models shown in this section are not very sophisticated, and their main purpose is to illustrate the application of our technique to concrete examples and to motivate an interpretation of some features obtained, like complex eigenvalues or the degeneration of dipole and quadrupole modes.

### 3.1 Spherical models

In order to verify our algorithm, we have applied it to the simple spherical \( \alpha ^2 \)-dynamo model of Krause and Steenbeck (1967), consisting of a sphere of radius \( R \) with constant \( \alpha \)-effect:

\[
\alpha (x) = \begin{cases} 
\alpha_0 & |x| < R \\
0 & |x| > R 
\end{cases} \quad \omega \equiv 0 . \tag{27}
\]

This model can be treated analytically. For \( \sigma \equiv \text{const} \), the critical values of the dynamo number \( C = \mu_0 \sigma \alpha_0 R \) are given by the roots of

\[
(n+1) \cdot C \cdot j_{n-1}(C) + n(2n+1) \cdot j_n(C) = 0 \quad ; n = 1, 2, \ldots \tag{28}
\]

with Spherical Bessel functions \( j_n(z) = \sqrt{\pi / 2z} J_{n+1/2}(z) \). The integer \( n \) is a kind of “mode number” in polar distance \( \vartheta \); odd \( n \) give modes with dipole\(^2 \), even \( n \) with quadrupole\(^2 \) symmetry.

In Table 1, the exact values from Equation (28) for the lowest six critical dynamo numbers (bottom line) are compared with numerical values for different grid sizes. The lowest critical dynamo number is \( C_{\min} \approx 3.506 \), which can be compared to the estimate (51) given in Section 4.3 that yields \( C_{\min} \geq 1 \) in this case.

Table 1 shows that a semi-quantitative picture of the distribution of critical eigenvalues is already obtained for quite coarse grids. For the finest grid applied here, the four leading dynamo numbers have a relative error less than 1.5%.

The error does not drop very quickly with grid refinement. Actually, it can be expected to decay like \( \Delta C = C_N - C_{\text{exact}} \sim \ln N / N \), while computation time (the number of floating point operations) is dominated by the matrix-eigenvalue algorithm, which is an \( N^3 \)-process: \( t_{\text{comp}} \sim N^3 \) for the full matrices involved here. The order \( \ln N / N \) is due to the very unsophisticated discretisation we adopted, simply replacing the integrals (21–25) over a cell \( [\bar{\vartheta}, \vartheta) \times [z_i - \delta z / 2, z_i + \delta z / 2] \) by the value at \( (\vartheta, z_i) \), multiplied by \( \delta \vartheta \delta z \). According to the weak singularity in (9), one has to handle the case \( (\vartheta, z) = (\bar{\vartheta}, z_i) \), \((\vartheta, z) \in [\bar{\vartheta}, \vartheta) \times \mathbb{R} \) separately. The simplest procedure is to drop

\(^2\)For the nomenclature ‘dipole – quadrupole’, we refer only to the poloidal field here. Since in the given example \( \alpha \) is symmetric with respect to the equatorial plane, the symmetry properties of poloidal and toroidal fields are opposite, not equal as in the more physical case of antisymmetric \( \alpha \). For a more rigorous definition of dipole and quadrupole modes in the case of these two kinds of symmetry, cf. Dobler and Rädler, 1997.
**Table 1:** Comparison of numerical values of the first six critical dynamo numbers \( C = \mu_0 \alpha \alpha_0 R \) for different grid sizes, with the exact result for the Krause-Steenbeck dynamo (27). \( N \) is the number of grid points within the dynamo region (here: within the sphere). The numbers in brackets in the last line are the mode numbers \( n \) from Equation (28). \( n = 1, 3, 5, \ldots \) gives dipole modes, \( n = 2, 4, 6, \ldots \) quadrupole modes.

<table>
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<td>7.9723</td>
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<tr>
<td>196</td>
<td>3.5535</td>
<td>5.0863</td>
<td>6.4838</td>
<td>6.6915</td>
<td>7.8232</td>
</tr>
<tr>
<td>394</td>
<td>3.5286</td>
<td>5.0348</td>
<td>6.3980</td>
<td>6.5933</td>
<td>7.7234</td>
</tr>
<tr>
<td>Theory:</td>
<td>3.5059(1)</td>
<td>4.9819(2)</td>
<td>6.3090(3)</td>
<td>6.5024(1)</td>
<td>7.5651(4)</td>
</tr>
</tbody>
</table>

This contribution, which introduces an error of order

\[
\int d\phi' d\zeta' \left[ \partial_i E^\phi(\phi, \phi', \zeta, \zeta') - \delta^\phi E^\phi(\phi, \phi', \zeta, \zeta') \right] \sim \delta \phi \delta \zeta \ln \delta \phi \sim \frac{1}{N} \ln N \tag{29}
\]

due to \( \hat{D} \); the operators \( \hat{E} \), \( \hat{F} \) give terms of the same order, while the integrands in \( \hat{A} \) and \( \hat{G} \) are not singular at all.

We used this simple scheme because it is also the most flexible one and does not require any special treatment of points near the surface of the dynamo region. Much more effective schemes could be implemented using higher order integration formulae, but then the points close to the boundary need special treatment (as they do in higher-order finite-difference schemes).

This numerical cost of our technique is difficult to compare with finite difference schemes where only sparse matrices are involved. Firstly because the latter are often used to solve the dynamo Cauchy problem which can yield only the dynamo number and growth rate of the fastest growing mode, but never gives an overview over the whole spectrum of eigenmodes and dynamo numbers. Secondly, the cost of embedding the dynamo region into a much larger volume is difficult to estimate in a general way. We would expect finite-difference methods to be more efficient when only the leading field mode is of interest, but high accuracy is needed. On the other hand, our method probably has a lead for not too high accuracy required.

Figure 2 shows the numerically obtained first eigenmode for the dynamo (27). This graphic very closely resembles the analytical solution; the biggest deviations from the exact result have been found around the \( z \)-axis, i.e. in a region of quite small volume. We emphasise that, unlike finite difference methods in cylindrical coordinates, our approach does not require additional (“inner”) boundary conditions to be posed on the vertical axis.
The next dynamo model we examined is more physical than Equation (27) in that the $\alpha$-effect is antisymmetric with respect to the equatorial plane:

$$\alpha(\mathbf{x}) = \begin{cases} 
\alpha_0 \cos \vartheta & , r < R \\
0 & , r > R 
\end{cases} \quad \omega \equiv 0 , \quad (30)$$

where $\cos \vartheta = z/\sqrt{r^2+z^2}$ is the cosine of the polar distance angle.

For surrounding vacuum, Roberts (1972) found the first two modes to be a dipole ($C_{\text{crit}} = 7.641$) and a quadrupole ($C_{\text{crit}} = 7.808$) one. Our model results for homogeneous conductivity are shown in Table 2. Here the first two, identical, dynamo numbers correspond to one dipole and one quadrupole mode. More generally, all eigenmodes appear in dipole-quadrupole pairs of equal critical dynamo number, a phenomenon we will refer to as dipole-quadrupole degeneration.

This degeneration is related to Roberts’ (1960) adjointness theorem and has been proven by Proctor (1977b, 1977c). It is not restricted to the special dynamo model (30), but rather appears in a broad class of kinematic dynamo systems. In a subsequent paper (Dobler and Rädler, 1997), we show that

---

3Strictly speaking, every linear combination of these two modes is an eigenfunction to the same eigenvalue, too, and choosing the modes with well-defined parity as basic states is arbitrary, but useful.
**Table 2:** Critical dynamo numbers for the dynamo (30). The dynamo numbers appear in pairs with one of them representing a dipole and the other a quadrupole mode. The two leading modes are shown in Figure 3.

<table>
<thead>
<tr>
<th>N</th>
<th>C</th>
<th>C</th>
<th>C</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>394</td>
<td>6.73345</td>
<td>6.73345</td>
<td>10.5839</td>
<td>10.5839</td>
</tr>
<tr>
<td></td>
<td>11.3624</td>
<td>11.3624</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Figure 3:* First dipole (left) and quadrupole mode (right) of the dynamo (30). Both modes correspond to a critical dynamo number \( C = 6.733 \).

a.) for \( \sigma \equiv \text{const} \) and \( \alpha \) antisymmetric with respect to the equatorial plane, any \( \alpha^2 \)-dynamo shows dipole-quadrupole degeneration

and

b.) if for an \( \alpha^2 \omega \)-dynamo in addition the velocity field \( u \) is mirror-symmetric with respect to the equatorial plane, then generalised dipole-quadrupole degeneration takes place, i.e. with every dipole mode to the velocity field \( u = \omega y e_x + u^p \) there exists a quadrupole mode to the reversed velocity field \( -u \), and vice versa.

Our proof is, in some sense, similar to what Proctor (1977b, 1977c) showed for a slightly different class of dynamos, and makes use of the integral-equation formalism introduced in this paper. It is only because our numerical scheme closely follows this formulation and conserves the relevant symmetry properties, that we have found exact degeneration numerically even on an arbitrarily coarse grid.

### 3.2 Elliptical models

We have examined an \( \alpha^2 \omega \)-dynamo in an oblate spheroid

\[
\frac{y^2}{a^2} + \frac{z^2}{b^2} < 1
\] (31)
for different aspect ratios \( a/b \in \{1, 3, 8\} \). Note that we used different grid spacings in vertical and horizontal direction to adapt the grid to the geometry in consideration.

For \( \alpha(x) \) and \( \omega(x) \) we chose

\[
\alpha(x) = \begin{cases} \alpha_0 \frac{z}{b} & ; \frac{b^2}{a^2} + \frac{z^2}{b^2} < 1 \\ 0 & , \text{ otherwise} \end{cases}
\]

(32)

\[
\omega(x) = \frac{C_\omega \alpha_0}{C_\alpha a^2} \begin{cases} (\varrho - a) & ; \frac{b^2}{a^2} + \frac{z^2}{b^2} < 1 \\ \text{linear (in } r) \text{ to zero} & , 1 < \frac{b^2}{a^2} + \frac{z^2}{b^2} < 4 \\ 0 & , \text{ otherwise} \end{cases}
\]

(33)

where the coefficients are chosen such that

\[
C_\alpha = \mu_0 \sigma |\alpha_{\text{max}}| a = \mu_0 \sigma_0 a, \quad |C_\omega| = \mu_0 \sigma \left| \frac{\partial \omega}{\partial \varrho} \right| \frac{a^3}{\alpha_{\text{max}}}.
\]

(34)

The second line in the definition of \( \omega \) means a linear interpolation in spherical radius \( r = \sqrt{\varrho^2 + z^2} \) from the value on the surface of the ellipsoid to value zero on the surface of an embedding, two times larger, ellipsoid. This was applied in order to avoid discontinuities in \( \omega(x) \) at the surface of the (inner) ellipsoid — a problem that never arises when the dynamo is surrounded by vacuum. There are, however, no really good arguments to proceed like this as long as \( \alpha(x) \) is still allowed to be discontinuous, because \( \alpha \) and \( \omega \) enter the induction equation (4) at the same level of differentiation. Thus, a discontinuity in \( \omega \) should not be more problematic than one in \( \alpha \), which is inherent to many dynamo models.

**Table 3:** Critical dynamo numbers for the \( a^2 \)-dynamo (32,33), \( C_\omega = 0 \) with different aspect ratios \( a/b \).

<table>
<thead>
<tr>
<th>( \frac{a}{b} )</th>
<th>( N )</th>
<th>( \frac{C_\alpha}{a^2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>344</td>
<td>18.2931 18.2931 25.3462 25.3462 32.3976 ( \pm 0.34277i ) 32.3976 ( \pm 0.34277i )</td>
</tr>
<tr>
<td>8</td>
<td>392</td>
<td>40.2809 40.2809 45.3049 ( \pm 4.1090i ) 45.3049 ( \pm 4.1090i ) 49.6823 49.6823</td>
</tr>
</tbody>
</table>

For every of the three ellipsoids, three different ratios \( |C_\omega/C_\alpha| \in \{0, 1, 5\} \) were examined. For \( C_\omega = 0 \), the \( a^2 \)-dynamo, we again have dipole-quadrupole degeneration, i.e. dipole and quadrupole modes have equal conditions of excitation, as can be seen in Table 3. The critical dynamo numbers grow quickly with the aspect ratio \( a/b \), which is mainly due to our choice of the semi-major axis \( a \) as the length scale for \( C_\alpha \). Especially for the very flat case \( a/b = 8 \), the “disc thickness” \( b \) is the more relevant scale as we know from models for the disc dynamo, where a sensible choice for the dynamo number is \( C_{\alpha,\text{disc}} = \mu_0 \sigma \alpha_s b \) (confer e.g. to Parker, 1971, Ruzmaikin...
et al., 1988). The complex dynamo numbers in Table 3 will be discussed in one of the following paragraphs.

Table 4 shows the critical dynamo numbers for $C_\omega/C_\alpha = \pm 1$. Now, differential rotation breaks the dipole-quadrupole degeneration. The generalised degeneration (item b. in Section 3.1), however, makes Table 4 valid for positive as well as negative ratio $C_\omega/C_\alpha$: For $C_\omega/C_\alpha = +1$, the first mode for all three aspect ratios is a dipole mode and is shown on the left half of Figures 4-6. Thus for $a/b = 3$, for example, we have a dipole mode with $C_\alpha = 17.4321$ (shown in Figure 5) and a quadrupole mode with $C_\alpha = 19.1533$ (not shown here). For $C_\omega/C_\alpha = -1$, we have the quadrupole mode shown in Figure 5 as first mode, with critical dynamo number $C_\alpha = 17.4321$ and the first dipole mode (not shown) corresponds to $C_\alpha = 19.1533$.

### Table 4: Critical dynamo numbers for the $\alpha^2\omega$-dynamo (32,33), $C_\omega/C_\alpha = \pm 1$ (i.e. $\omega = \pm (\rho - a)$) with different aspect ratios $a/b$. The first mode (for all three aspect ratios) is a dipole mode (shown in the left half of Figures 4-6) for sign “+” and a quadrupole mode (right half of Figures 4-6) for sign “−”.

<table>
<thead>
<tr>
<th>$\frac{a}{b}$</th>
<th>$N$</th>
<th>$C_\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>392</td>
<td>39.3091 41.4539</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Figure 4:** The first mode for the “ellipsoid” dynamo (32,33) for $a=b=1$. Left half: $C_\omega/C_\alpha = +1$; right half $C_\omega/C_\alpha = -1$.

The complex dynamo numbers in Tables 3–5 probably represent oscillating modes. In fact, as discussed in Section 4.2, an oscillating mode will show itself as mode with complex dynamo number, and only by adjusting $\Omega = \Im \gamma$ to the oscillation frequency in Equation (45) below, one gets a real dynamo number. On the other hand, it is not clear whether an arbitrary complex dynamo number really has such a frequency $\Omega$, for which its imaginary part vanishes. Thus, some of the complex
dynamo numbers, but not necessarily all of them, represent the oscillating modes of the system. In any case, the ordering in Tables 3–5 (by modulus $|C_n|$) is arbitrary as for the complex dynamo numbers, and only the procedure outlined in Section 4.2 will clarify their position.

For the axisymmetric modes of an $\alpha^2\omega$-dynamo, complex dynamo numbers always appear in complex conjugate pairs, because we solve a real eigenvalue problem (having a characteristic equation with real coefficients). Physically, such a conjugate complex pair of eigenvalues represents only one oscillating magnetic field mode (if one at all), and not a pair of modes. Similarly, the “complex conjugate quadruple” of eigenvalues in Table 3 represents one dipole and its adjoint quadrupole mode. This argumentation does not apply to non-axisymmetric modes, because the ansatz $B_{\theta/\varphi/z}(\mathbf{x}) = B_{\theta/\varphi/z} e^{im\varphi}$ introduces a true complex component into the eigenvalue
Table 5: Same as Table 4, but for $C_\omega/C_\alpha = \pm 5$ (i.e. $\omega = \pm 5(a - \alpha)$).

<table>
<thead>
<tr>
<th>$\frac{a}{\alpha}$</th>
<th>$N$</th>
<th>$C_\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>366</td>
<td>9.00838 10.9131 12.9290 17.1134 17.4773 17.687</td>
</tr>
<tr>
<td>3</td>
<td>344</td>
<td>14.5274 21.9044 22.0822 29.6684 31.7615 32.7964 $\pm 4.22042i$</td>
</tr>
<tr>
<td>8</td>
<td>392</td>
<td>32.3494 $\pm 1.3272i$ 42.6577 46.6963 $48.9744 \pm 0.255348i$ 51.7960 $\pm 1.7490i$ 51.3967</td>
</tr>
</tbody>
</table>
quantum mechanics for the (non-hermitian) differential operator $\hat{D}$ from the induction equation (4)

$$\hat{D}B := \nabla(\alpha B + u \times B - \beta \nabla B) ,$$  \hspace{1cm} (35)

i.e. probably localised magnetic modes correspond to discrete eigenvalues (growth rates) $\gamma$, while the continuous spectrum is caused by non-localised modes. Whenever we use in the following sections cautious formulations like “we can expect” or “the continuum should be on the negative half axis”, we argue on the base of this analogy and the corresponding results are not mathematically strict but only strongly motivated; this is in contrast to the strict results on boundedness and compactness for the integral operators in Equations (13) and (45) below.

As we will see in this section, the derivation of our integral equation (9) or (45) implies that the magnetic field modes $B(x)$ are localised. This means that in the integral-equation formalism non-localised fields and, if the analogy to quantum mechanics holds, the continuous spectrum are discarded from the very start (but only for growth rates $\gamma$ that are not real negative, see below). As Meinel shows, these modes always decay with time and are thus of minor interest in dynamo theory. It is only by excluding these non-localised modes, that some severe problems connected with the differential operator from the induction equation are overcome, and an integral-equation description of the dynamo problem is possible.

### 4.2 Time dependent magnetic field

The most general integral equation for the time-dependent case is obtained from Equation (4) (but in dimensional form) by applying Green’s function of the heat-conduction operator on the left:

$$G(x-x', t) = \frac{1}{(4\pi \tau t)^{3/2}} e^{-\frac{(x-x')^2}{4\tau t}}$$  \hspace{1cm} (36)

with $\eta = 1/(\mu_0 \sigma) = const.$; again, variations in $\sigma(x)$ are transmuted into $\beta(x)$, cf. Section 2.1. The corresponding formula can be found in Rädler (1968) or the book of Krause & Rädler (1980) and integration by parts transforms it into

$$B(x, t) = \int dx^3 G(x-x', t) B_0(x')$$

$$- \int_0^t \int dx^3 \nabla G(x-x', t-t') \times F(x', t'),$$  \hspace{1cm} (37)

where $B_0(x) := B(x, t=0)$ is the initial field. Although important as an analytical tool, this equation is too general for numerical applications and we will now turn to a more particular formulation.

If we introduce the ansatz

$$B(x, t) = \bar{B}(x) e^{\gamma t}, \hspace{1cm} \gamma \text{ complex},$$  \hspace{1cm} (38)
into the induction equation (4) and omit the tilde, we get

$$\frac{1}{C}(\Delta B - \gamma B) = -\text{curl}\mathcal{F}, \quad \text{div} B = 0,$$  \hspace{1cm} (39)

with \(\mathcal{F}\) given by (3). For a given dynamo system, \(C\) is fixed and Equation (39) is an eigenvalue problem with eigenvalues \(\gamma\). Formally, we can as well fix \(\gamma\) and get a generalised eigenvalue problem with eigenvalues \(1/C\) which will in general be complex. As only real dynamo numbers are physically meaningful, one has to choose such values of \(\gamma\) for which one of the \(C\) is real.

Equation (39) is of the type of the generalised inhomogeneous Helmholtz equation

$$\Delta A - \gamma A = q(x).$$  \hspace{1cm} (40)

This becomes a Helmholtz equation in the literal sense, describing undamped standing waves, only for real and negative \(\gamma\), a case we will exclude later for several reasons.

Let \(K_R = \{ x \mid x^2 \leq R^2 \}\) be a sphere of radius \(R\). For the boundary condition \(A(x) = 0\) on \(\partial K_R\), Equation (40) can be solved for \(A\) by means of the corresponding Green's function \(G_R(x,x')\):

$$A(x) = \int_{K_R} G_R(x,x') q(x') \, dx'^3.$$  \hspace{1cm} (41)

Note that \(G_R\) is unique only if \(\gamma\) is not on the negative real axis, \(\gamma \notin \mathbb{R}^-\), and we will restrict ourselves here and in the following to that case (cf. Appendix A.2). Accordingly, for the boundary condition \(B(x) = 0\) on \(\partial K_R\), Equation (39) can be written as

$$B(x) = -C \int_{K_R} G_R(x,x') \text{curl}' \mathcal{F} \, dx'^3$$ \hspace{1cm} (42)

$$= C \int_{D} \text{grad}' G_R(x,x') \times \mathcal{F} \, dx'^3$$ \hspace{1cm} (43)

for any sphere \(K_R\) that is large enough to enclose \(D\).

For \(R \to \infty\), Green's function of Equation (40) is well known:

$$G_\infty(x,x') = G_\infty(x-x') = -\frac{1}{4\pi} \frac{e^{-\sqrt{\gamma}|x-x'|}}{|x-x'|};$$  \hspace{1cm} (44)

here and in the following, the root \(\sqrt{\gamma}\) of a complex number is chosen such that \(\Re\sqrt{\gamma} \geq 0\). The boundary condition is in this case simply that \(B(x) \to 0\) for \(|x| \to \infty\), and we get (again in dimensionless form)

$$\left( \frac{A(x)}{B(x)} \right) = \int \frac{(x-x') \times \mathcal{F}}{|x-x'|^3} e^{-\sqrt{\gamma}|x-x'|} \left(1 + \sqrt{\gamma}|x-x'|\right) \, dx'^3 = -\frac{4\pi}{C} B(x).$$  \hspace{1cm} (45)
Like in Section 2.1 it is easily shown that the operator on the left hand side of (45) is compact in the case of homogeneous conductivity (and thus bounded) for \( \gamma \notin \mathbb{R}^{-} \).

For a given value of \( \gamma \), Equation (45) is again an integral-eigenvalue equation for \( B(x) \) with eigenvalues \(-4\pi/C\). In order to get numerically the modes with a given growth rate \( \Re \gamma \) and the corresponding dynamo numbers (e.g. \( \Re \gamma = 0 \) and \( C = C_{\text{crit}} \)) one has to solve Equation (45) numerically for different values of \( \Omega := 2\pi \gamma \), which yields complex functions \( C(\Omega) \) that are continuous because they are eigenvalues of an integral operator that depends smoothly on \( \gamma \) (and so does the matrix obtained by simple discretisation).

One has to find the zeros of the imaginary part of these functions, \( \Im C(\Omega) \geq 0 \), because only real dynamo numbers are physically meaningful. The corresponding value of \( \Omega \) is then the oscillation frequency of the time dependent mode.

For \( \gamma \neq 0 \), \( \gamma \notin \mathbb{R}^{-} \), we have \( \Re \sqrt{\gamma} > 0 \), and \( B(x) \) decays for \( r := |x| \to \infty \) exponentially according to

\[
B(x) = \frac{-\mu_{0}v_{L}}{4\pi} e^{-\sqrt{\pi} \sqrt{\frac{\gamma}{k}}} \left( \frac{1}{r} + O\left( \frac{1}{r^{2}} \right) \right) \frac{x}{r} \times LI_{1}
\]

with \( LI_{1} = \sigma \int \mathcal{F}(x') \, dx' \).

For \( \gamma = 0 \) we retrieve Equation (9) and the field decays like \( \mathcal{O}(1/r^{2}) \). In both cases we have \( \|B\|_{2} < \infty \). Thus, only requiring \( \gamma \) to be not real negative and the fields \( B(x) \) to vanish on a boundary \( \partial \mathcal{K}_{R} \) that was then shifted to infinity, we find that all eigenfunctions \( B(x) \) are moreover square integrable. That is, they are localised in space and we can expect the spectrum \{\gamma\} for a given dynamo number \( C \) to be discrete anywhere off the negative real half axis.

In an infinite conductor, the equation \( \eta \Delta B - \gamma B = 0 \) for the free decay of magnetic field has a continuous spectrum \( \gamma = -\eta k^{2}, \ k \in \mathbb{R}^{3} \) and the corresponding eigenfunctions \( B \sim j_{l}(k r) \cdot Y_{l}^{m} (\theta, \varphi) \) tend to zero for \( r \to \infty \) (at least they can be chosen so), but are not localised. We can expect the dynamo equation (4) to have such a continuum, too, and since only for real and negative \( \gamma \) modes with \( B \to 0 \) for \( r \to 0 \) may be non-localised, the continuum should still be on the negative half axis.

### 4.3 A necessary condition for magnetic field generation

We already mentioned that the integral operators in Equations (9) and (45) are bounded for \( \beta \equiv 0 \). It is not difficult to derive a concrete upper bound for the norm \( \|A^{(\gamma)} \| \).

Let \( \|u\|_{\infty} = \max_{x \in \mathcal{D}} |u(x)| \) and \( \|g\|_{\infty} = \max_{x \in \mathcal{D}} \|g(x)\|_{2} \) (where \( \|g\|_{2} \) is the spectral norm, or any other matrix norm that is compatible with the Euclidean vector norm) denote the maximum norms of the vector function \( u(x) \) and the tensor function \( \alpha(x) \); let \( L_{P} \) the diameter of the smallest sphere enclosing the dynamo region \( \mathcal{D} \). Noting that \( 0 < e^{-\sqrt{\gamma} |x|} \left( 1 + \sqrt{\gamma} |x| \right) \leq 1 \quad \forall x \), we find for an arbitrary bounded vector function \( b \) in the dimensionless units from (4) and (9):

\[
\left| A^{(\gamma)} b(x) \right| \leq \int_{\mathcal{D}} \left| \frac{x-x'}{|x-x'|^{3}} \cdot |u(x') \times b(x') + \alpha(x') b(x')| \right| \, dx'^{3}
\]
are now defined by
\[ \frac{dx^3}{|x-x'|^2} \cdot \left( \| u \|_\infty + \| \alpha \|_\infty \right) \cdot \| b \|_\infty \] (48)
\[ \leq 2 \pi L_D \cdot \left( \| u \|_\infty + \| \alpha \|_\infty \right) \cdot \| b \|_\infty . \] (49)

In other words, \( \hat{A}^{(\gamma)} \) is bounded, if \( D \) is and \( \alpha \) and \( u \) are, and then
\[ \| \hat{A}^{(\gamma)} \| \leq 2 \pi L_D \left( \| u \|_\infty + \| \alpha \|_\infty \right) . \] (50)

It is essential here that our operator acts on functions on a bounded region \( D \), because on unbounded regions the maximum modulus \( \max_x |f(x)| \) of a function \( f(x) \) is not a norm of \( f \).

The same argumentation as above is valid in the case of \( \alpha \)-quenching, provided that the norms of \( u \) and \( \alpha \) are now defined by \( \| u \|_\infty = \max_x |u(x; \alpha; B)|, \| \alpha \|_\infty = \max_{x, B} \| \alpha(x; \alpha; B) \|_2 \) and are finite.

As the eigenvalues of a linear operator are bounded by the operator norm (cf. Kress, 1989), we derive from the upper bound (50) as a corollary an estimate for the dynamo numbers \( C \) of steady (linear or nonlinear) and time dependent (with \( \gamma \notin \mathbb{R}^- \)) dynamos with homogeneous conductivity \( (\sigma = \text{const}) \):
\[ |C| = \mu_0 \sigma L |U| \geq \frac{2L}{L_D} \frac{U}{\| u \|_\infty + \| \alpha \|_\infty} . \] (51)

This is a necessary condition for the excitation of magnetic field by mean-field dynamos with homogeneous conductivity. It has first been derived by Roberts (1967, 1994) for the steady case. Other necessary conditions have been given by Backus (1958) and Childress (1969). Backus derives a relation between the rate-of-strain tensor \( \sigma_{ik} := (\partial_i u_k + \partial_k u_i)/2 \) and the lowest free decay rate \( \Gamma_{\min} \) for the given geometry. His condition applies to vacuum surrounding the dynamo and was improved and generalised to a dynamo embedded in a conducting medium (although supposing that \( \sigma(x) = O(1/r) \) for \( r \to \infty \)) by Proctor (1977a).

The condition of Childress (1969) is more similar to ours in that it gives an estimate for the magnitude of the velocity field itself. For a spherical dynamo with constant electrical conductivity and solenoidal motions, it reads
\[ \mu_0 \sigma R \| u \|_\infty \geq \mu_0 \sigma R \frac{\Delta_{\max} u}{2} \geq \frac{\pi}{2} , \] (52)
where \( \Delta_{\max} u \) is the maximum relative velocity inside the sphere. For a sphere, our estimate (51) yields the weaker result \( |C_{\text{crit}}| = \mu_0 \sigma R \| u \|_\infty \geq 1 \).

For a cylinder of height \( 2h \) and radius \( R \), the “geometric integral” in (48) can also be given explicitly and we get
\[ |C| \geq \frac{L}{h} \left( \ln \frac{R}{h} + 1 + \frac{h}{2R} \right) \| u \|_\infty + \| \alpha \|_\infty ; \] (53)
for a thin disc, the term $h/(2R)$ can be neglected.

As (51) holds for time dependence $\sim \exp(\gamma t)$ with $\gamma \notin \mathbb{R}^-$, we can conclude directly that for dynamo numbers $C$ below this lower bound only decay of the magnetic field modes is possible and that this decay occurs with $\gamma \in \mathbb{R}^-$, i.e. non-oscillatory, as otherwise a contradiction to the condition (51) would occur. Nothing similar can be said when the dynamo number is above the bound (51), but below the lowest critical value.

4.4 Variable conductivity

When the function $\beta(x)$ is different from zero on a positive volume, the integral operator $\mathbb{A}$ from Equation (9) is no longer compact. This is not very surprising, as we apply the integral operator “$\text{curl}^{-1}$” $= \int \text{grad}' \frac{1}{|x-x'|} \cdot dx'^3$ to the curl $\text{curl}'\mathbf{B}$ and the identity operator is bounded, but not compact (cf. Kress, 1989). More insight can be gained by eliminating the differentiation of $\mathbf{B}$, integrating (9) by parts according to

$$
\int_{\mathbb{V}} a \times \text{curl}'\mathbf{b} \, dx'^3 = - \int_{\mathbb{V}} a \times [b \times df'] + \int_{\mathbb{V}} (b \, \text{div}'a - b\nabla a) \, dx'^3, \tag{54}
$$

with $(b\nabla a)_i := b_j \partial_j a_j$, and using the well-known formula from potential theory

$$
\text{div}' \frac{x-x'}{|x-x'|^3} = -4\pi \delta(x-x'). \tag{55}
$$

We thus get the following integral equation in non-dimensional form

$$
\int \left\{ \frac{(x-x') \times [\alpha \mathbf{B} + \mathbf{u} \times \mathbf{B} + \text{grad}' \beta \times \mathbf{B}]}{|x-x'|^3} + \beta(x') \left( \frac{\mathbf{B}}{|x-x'|^3} - 3 \frac{(\mathbf{B} \cdot (x-x')) (|x-x'|)}{|x-x'|^5} \right) \right\} \, dx'^3 + 4\pi \beta(x) \mathbf{B}(x) = -\frac{4\pi}{C} \mathbf{B}(x) \tag{56}
$$

(note that the term $\text{grad}' \beta \times \mathbf{B}$ does not represent turbulent diamagnetism, as the latter is included in $\mathbf{u} \times \mathbf{B}$ from the very start).

This is again a kind of eigenvalue equation for $\mathbf{B}(x)$ on $\mathcal{D}$ with eigenvalues $-4\pi/C$. There are, however, two facts that make its mathematical and numerical treatment much more complicated than that of Equation (9). First, the term $\mathbb{A}_I \mathbf{B} := 4\pi \beta(x) \mathbf{B}(x)$, like the identity operator on function spaces, is bounded, but not compact (cf. Kress, 1989, theorem 2.19). Hence, Equation (56) is no longer an eigenvalue problem for a compact operator and, deprived of the power of Riesz theory, we no longer can exclude the existence of a continuous spectrum.

Second, the kernel of the second integral operator $\mathbb{A}_II$

$$
(\mathbb{A}_II \mathbf{B})(x) := \int \beta(x') \left( \frac{\mathbf{B}(x')}{|x-x'|^3} - 3 \frac{(\mathbf{B}(x') \cdot (x-x')) (x-x')}{|x-x'|^5} \right) \, dx'^3 \tag{57}
$$
is singular (not only weakly singular as above) and the integral exists only as a principal value. $A_1$ is even unbounded on spaces of continuous functions, but bounded on Hölder spaces. In any case it is not a compact operator.

Probably, an integral equation involving a Green's function for the given conductivity distribution would again give us a compact operator. But even for the simple case of a sphere of constant conductivity in vacuum, there seems to be no closed analytical expression for the corresponding Green's function and it can only be given by an infinite series in special functions (cf. Bräuer & Rädler, 1987).

Of course, these questions should be investigated in more detail. It may be possible to overcome the difficulties, but the numerical solution of (56) can prove to be much less straightforward than in the case of homogeneous conductivity.

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References


Appendix

A.1 The integrals $E^m_{1/c/s}$

In Section 2.3 we introduced three symbols $E^m_1, E^m_c, E^m_s$, that are defined as follows:

$$E^m_1 = E^m_1(\varphi, \varphi', z-z') = 2 \int_0^\pi \frac{\cos m\varphi'}{[\varphi^2 + \varphi'^2 - 2\varphi\varphi' \cos \varphi' + (z-z')^2]^{3/2}} d\varphi'$$  \hspace{1cm} (58)

$$E^m_c = E^m_c(\varphi, \varphi', z-z') = 2 \int_0^\pi \frac{\cos \varphi' \cos m\varphi'}{[\varphi^2 + \varphi'^2 - 2\varphi\varphi' \cos \varphi' + (z-z')^2]^{3/2}} d\varphi'$$  \hspace{1cm} (59)

$$E^m_s = E^m_s(\varphi, \varphi', z-z') = 2i \int_0^\pi \frac{\sin \varphi' \sin m\varphi'}{[\varphi^2 + \varphi'^2 - 2\varphi\varphi' \cos \varphi' + (z-z')^2]^{3/2}} d\varphi'$$  \hspace{1cm} (60)

These integrals can be expressed in terms of complete elliptic integrals, but as far as we see, this results in no general expressions that apply to arbitrary $m$.

Alternatively, we can express the integrals in terms of hypergeometric functions. Formula 30 in §5 of Oberhettinger’s (1957) table of Fourier transforms, in connection with a limiting procedure and some standard multiplication theorems for trigonometric functions, can be used to obtain the following representation for $m=0$:

$$E^0_1(\varphi, \varphi', \zeta) = \frac{v_0}{2} _2F_1\left(-\frac{1}{2}, -\frac{1}{2}; 1; e^{-2a}\right)$$  \hspace{1cm} (61)

$$E^0_c(\varphi, \varphi', \zeta) = \frac{3}{4} v_0 e^{-a} _2F_1\left(-\frac{1}{2}, \frac{1}{2}; 2; e^{-2a}\right)$$  \hspace{1cm} (62)

$$E^0_s(\varphi, \varphi', \zeta) = 0 ,$$  \hspace{1cm} (63)

and for $m>0$:

$$E^m_1(\varphi, \varphi', \zeta) = v_m (m+\frac{1}{2}) _2F_1\left(-\frac{1}{2}, m-\frac{1}{2}; m+1; e^{-2a}\right)$$  \hspace{1cm} (64)

$$E^m_c(\varphi, \varphi', \zeta) = \frac{v_m}{2} \left[ me^a _2F_1\left(-\frac{1}{2}, m-\frac{3}{2}; m; e^{-2a}\right) + \frac{(m+\frac{1}{2})(m+\frac{3}{2})}{m+1} e^{-a} _2F_1\left(-\frac{1}{2}, m+\frac{1}{2}; m+2; e^{-2a}\right) \right]$$  \hspace{1cm} (65)

$$E^m_s(\varphi, \varphi', \zeta) = \frac{v_m}{2} \left[ me^a _2F_1\left(-\frac{1}{2}, m-\frac{3}{2}; m; e^{-2a}\right) - \frac{(m+\frac{1}{2})(m+\frac{3}{2})}{m+1} e^{-a} _2F_1\left(-\frac{1}{2}, m+\frac{1}{2}; m+2; e^{-2a}\right) \right] .$$  \hspace{1cm} (66)

Here the abbreviations

$$a = \text{arccosh} \frac{\varphi^2 + \varphi'^2 + \zeta^2}{2\varphi\varphi'} = \text{arsinh} \sqrt{\frac{(\varphi^2 - \varphi'^2)^2 + 2(\varphi^2 + \varphi'^2)\zeta^2 + \zeta^4}{2\varphi\varphi'}}$$  \hspace{1cm} (67)

$$v_m = \frac{\pi e^{-a(m-\frac{1}{2})}}{(\varphi\varphi')^{3/2} \sinh^2 a} \frac{(2m)!}{2^{2m} (m!)^2} = \frac{\sqrt{\pi} e^{-a(m-\frac{1}{2})}}{(\varphi\varphi')^{3/2} \sinh^2 a} \frac{(m-\frac{1}{2})!}{m!}$$  \hspace{1cm} (68)
have been used, and
\[
2F_1(a, b; c; x) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{x^n}{n!} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{x^n}{n!}
\]  
(69)
denotes Gauss' hypergeometric function (cf. Abramowitz and Stegun, 1980).

A.2 About the Helmholtz equation

The equation
\[
\Delta A - \gamma A = q
\]  
(70)
describes standing waves in an absorbing (for \( \gamma \notin \mathbb{R}^- \)) medium in the presence of sources \( q(x) \). In the special case \( \gamma = -k^2 \), \( k^2 = \omega^2/c^2 \in \mathbb{R}^+ \) this is the actual Helmholtz equation, describing standing waves of frequency \( \omega \) and wave number \( k \) in a non-absorbing medium with phase velocity \( c \).

The corresponding homogeneous problem, i.e. the homogeneous equation
\[
\Delta A - \gamma A = 0
\]  
(71)
with homogeneous boundary conditions \( A|_{\partial V} = 0 \) imposed on the boundary of a compact region \( V \) has the unique solution \( A \equiv 0 \), if \( \gamma \notin \mathbb{R}^- \). This can be shown by multiplying Equation (71) with \( A^* \), the complex conjugate of \( A \), and integrating over \( V \), which yields, after applying Green's integral theorem,
\[
\int_V \left( |\text{grad} A|^2 + \gamma |A|^2 \right) \, dx^3 = 0 .
\]  
(72)

If \( \text{Im} \gamma \neq 0 \) the imaginary part of Equation (72) yields \( A \equiv 0 \); if \( \gamma \) is real but non-negative, the real part of Equation (72) gives the same result (for \( \gamma = 0 \) this is a well-known property of the Laplace equation). Only for \( \gamma \in \mathbb{R}^- \) we cannot show the homogeneous problem to have a unique solution. With the physical interpretation of standing waves in mind, these results can be well understood: In the absence of sources, the only stationary solution of the wave equation in an absorbing medium is zero; for non-absorbing cavities, however, we know a whole spectrum of resonant wave solutions to exist, so no uniqueness can be expected in this case.

(Non-)uniqueness of the solution of the homogeneous problem directly implies (non-)uniqueness of Green's function \( G(x, x') \):
\[
\Delta G_V - \gamma G_V = \delta(x-x') , \quad G_V|_{\partial V} = 0 .
\]  
(73)

If we shift the boundary of \( V \) to infinity, Fourier techniques easily yield the corresponding Green's function \( G_\infty(x, x') \):
\[
G_\infty(x, x') = -\frac{2}{4\pi^2|x|} \int_0^{\infty} \frac{k \sin(k|x-x'|)}{k^2 + \gamma} \, dk = -\frac{1}{4\pi |x-x'|} e^{-\sqrt{\gamma} |x-x'|} \quad \forall \gamma \notin \mathbb{R}^- .
\]  
(74)
The other formal solution of (73),
\[ \tilde{G}_\infty(x) = -\frac{1}{4\pi |x|} e^{+\sqrt{\gamma}|x|} \]
(75)
can be discarded, because for \( \gamma \notin \mathbb{N}^+ \), \( \sqrt{\gamma} \) always has a positive real part, and this solution grows unboundedly for \( |x| \to \infty \).

For \( \gamma = -k^2 \), \( k^2 \in \mathbb{R}^+ \), the functions \( G_\infty, \tilde{G}_\infty \) turn to
\[ -\frac{1}{4\pi |x|} e^{\pm ik|x|} + A_{\text{hom}}(x) \]
(76)
where \( A_{\text{hom}}(x) = \sum \lambda_{l,m} j_l(kr) Y_l^m(\vartheta, \varphi) \) is a solution of the homogeneous equation (71). The function (76) is evidently no longer square integrable.