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On self–killing and self–creating dynamos

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Numerical studies with a spherical dynamo model have shown two remarkable phenomena. The model consists of a spherical body of an electrically conducting incompressible fluid surrounded by free space. In addition to a rotation of the body an inner motion due to a given forcing is considered which satisfies a no–slip condition at the boundary. The full interaction of magnetic field and motion is taken into account. Starting from a fluid motion capable of dynamo action and a very weak magnetic field it was observed that the growing magnetic field destroys the dynamo property of the motion and then decays, and that the system ends up in a state with another motion incapable of dynamo action and zero magnetic field. In another case with a motion unable to prevent small magnetic fields from decay it proved to be possible that stronger magnetic fields deform it so that a dynamo starts to work which enables the system to approach a steady state with a finite magnetic field.

Key words: Magnetohydrodynamics: nonlinear dynamo – dynamically consistent dynamo models – hydrodynamic bifurcation

1. Introduction

In many cases kinematic dynamo models developed in view of cosmic bodies have been extended by including the back–reaction of the magnetic field on the fluid motion. The simplest effect expected from introducing this kind of nonlinearity is the limitation of the growth of the magnetic field. There are, however, quite a few other effects such as changes in the dominating symmetry properties of magnetic field and motion or transitions to oscillatory, intermittent or chaotic behaviours. Remarkably enough, as we want to exemplify in this paper, the back–reaction of the magnetic field can also change the fluid motion resulting from a given distribution of forces such that it looses its dynamo capability and the magnetic field decays and cannot grow again.

Recently several investigations have been carried out on the possibility that a fluid motion which is stable with respect to non–magnetic perturbations and incapable of dynamo action can be destabilized by a finite magnetic field, and that then motion and magnetic field evolve toward a dynamo state. In particular the so–called Balbus–Hawley instability (Velikhov 1959, Chandrasekhar 1960, Balbus and Hawley 1991, 1992, 1998) has been considered in this sense (Brandenburg et al. 1995, Drecker et al. 1999). In the following we will describe another example in which a fluid motion maintained by given forces and lacking dynamo capability gains this property in an evolution initialized by a sufficiently strong magnetic field which then ends up with a steady dynamo.

2. The model

We consider a spherical body of an electrically conducting incompressible fluid in non–conducting surroundings. It is assumed that the magnetic flux density, $\boldsymbol{B}$, and the fluid velocity, $\boldsymbol{u}$, inside the body are governed by

$$\frac{\partial \boldsymbol{B}}{\partial t} = \eta \Delta \boldsymbol{B} + \mathrm{curl}\,(\boldsymbol{u} \times \boldsymbol{B}), \quad \mathrm{div}\,\boldsymbol{B} = 0\,, \tag{1}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla P + \nu \Delta \boldsymbol{u} - 2\,\boldsymbol{\Omega} \times \boldsymbol{u} + \frac{1}{\mu\rho}\,\mathrm{curl}\;\boldsymbol{B} \times \boldsymbol{B} + \boldsymbol{\mathcal{F}}, \quad \mathrm{div}\;\boldsymbol{u} = 0\,. \tag{2}$$

P is a modified pressure, and $\boldsymbol{\mathcal{F}}$ stands for an external body force. We refer to a rotating frame with $\boldsymbol{\Omega}$ being the angular velocity that defines the Coriolis force. As usual, η is the magnetic diffusivity, ν the kinematic viscosity, ρ the mass density and μ the magnetic permeability, all considered as constant. According to the assumption on non–conducting surroundings $\boldsymbol{B}$ has to continue as a potential field in outer space. Furthermore, the no–slip condition $\boldsymbol{u} = \boldsymbol{0}$ is assumed at the boundary. We use a spherical coordinate system (r, ϑ, φ) in the following such that its origin $r = 0$ coincides with the center of the fluid body and the axis $\vartheta = 0$ with the rotation axis. R denotes the radius of the body.

We specify the force $\boldsymbol{\mathcal{F}}$ so that equations (2) for vanishing rotation and magnetic field, that is $\boldsymbol{\Omega} = \boldsymbol{B} = \boldsymbol{0}$, allow a steady solution $\boldsymbol{u}$ with a flow pattern similar to that used in the kinematic dynamo studies by Pekeris, Accad and Shkoller (1973). Supposing symmetry of $\boldsymbol{u}$ about both the axis $\vartheta = 0$ and the equatorial plane $\vartheta = \pi/2$ we define it by

$$\begin{aligned} &\boldsymbol{u} = U_0 \boldsymbol{u}_0 , \quad \boldsymbol{u}_0 = \frac{1}{\sqrt{2}}(\boldsymbol{u}_0^{\mathrm{P}} + \boldsymbol{u}_0^{\mathrm{T}}) , \\ &\boldsymbol{u}_0^{\mathrm{P}} = -N^{\mathrm{P}} \frac{1}{\xi}\Big(f^{\mathrm{P}}(\xi)(3\cos^2\vartheta - 1)\,\boldsymbol{e}_r + \frac{d}{d\xi}\big(\xi f^{\mathrm{P}}(\xi)\big)\cos\vartheta\sin\vartheta\,\boldsymbol{e}_\vartheta \Big) , \qquad \boldsymbol{u}_0^{\mathrm{T}} = -N^{\mathrm{T}} f^{\mathrm{T}}(\xi)\sin\vartheta\,\boldsymbol{e}_\varphi , \\ &f^{\mathrm{P}}(\xi) = \mathrm{j}_2(\mu_{33}\,\xi) - \mathrm{j}_2(\mu_{33})\,\xi^2 , \qquad f^{\mathrm{T}}(\xi) = \mathrm{j}_1(\mu_{13}\,\xi) . \end{aligned} \tag{3}$$

U_0 is a constant with the dimension of a velocity and $\boldsymbol{u}_0^{\mathrm{P}}$ and $\boldsymbol{u}_0^{\mathrm{T}}$ denote poloidal and toroidal fields, respectively. N^{P} , N^{T} are positive constants such that the r.m.s. values of $\boldsymbol{u}_0^{\mathrm{P}}$ and $\boldsymbol{u}_0^{\mathrm{T}}$ and, consequently, of $\boldsymbol{u}_0$ are equal to unity. Furthermore, ξ is the fractal radius, $\xi = r/R$, the j_n are spherical Bessel functions of the first kind and the $\mu_{n\,l}$ their positive zeros, $\mu_{13} = 10.904122$ and $\mu_{33} = 13.698023$. We denote this solution of (2) by (o); the flow pattern is shown in Fig. 1.

We may express the intensity of the forcing $\boldsymbol{\mathcal{F}}$ by the magnitude U_0 of the velocity which it is able to maintain, or by the corresponding Reynolds number $U_0 R/\nu$. In that sense we introduce a forcing parameter $F = U_0 R/\nu$.

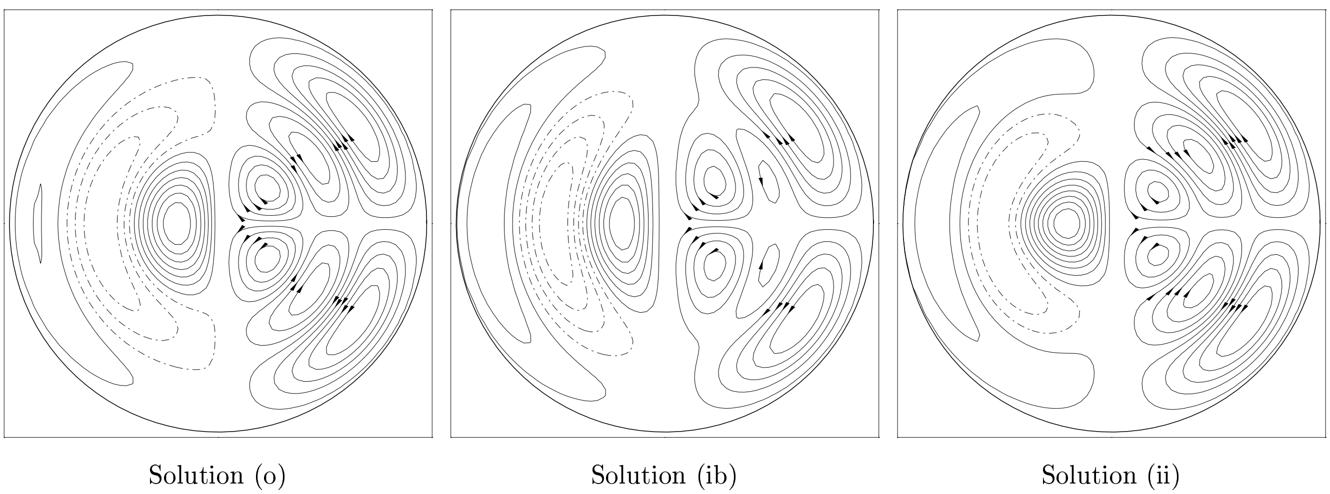

Fig. 1: The flow patterns of the solutions of (2). Solution (o) is given by (3), for (ib) and (ii) see Section 3. For each picture: right half — streamlines of the poloidal (i.e. meridional) part, left half — isolines of the toroidal (i.e.azimuthal) part with solid lines indicating flow out of, broken lines flow into the paper plane.

Models similar to that described here have been considered in earlier papers (Fuchs et al 1996, 1997). For the numerical investigation of our equations (1) and (2) we used the code explained there.

3. Fluid motions in the absence of magnetic fields

The solutions of (2) for $\boldsymbol{\Omega} = \boldsymbol{B} = \boldsymbol{0}$ as given by (3) proved to be stable only for $F < F_{\mathrm{crit}} = 15.4$. At $F = F_{\mathrm{crit}}$ a bifurcation occurs. For $F > F_{\mathrm{crit}}$ there are stable solutions deviating from (3).

Admitting now rotation, we characterize the magnitude of the Coriolis force by the Taylor number $Ta = (2\Omega R^2/\nu)^2$. For all following considerations we choose a forcing with $F = 17.27$. With $Ta = 10$ two stable steady solutions $\boldsymbol{u}$ have been found, which differ only slightly in their total kinetic energies but more in the distribution of the energy on the poloidal and toroidal motions, but for $Ta = 1000$ one solution only. We denote them by (ia), (ib) and (ii), respectively. In all cases $\boldsymbol{u}$ is again symmetric about both the axis of rotation and the equatorial plane.

In Table 1 some properties of these solutions are listed. For comparison the unstable solution (o) is included too. For all cases the Reynolds number $Re = UR/\nu$ is given with U being the r.m.s. value of $\boldsymbol{u}$. The flow patterns of the solutions (o), (ib) and (ii) are shown in Fig. 1; the pattern of (ia) was dropped since it is visually indistinguishable from that of (o).

Solution	Ta	Re	E_K^P/E_K^T	$\|h\|$	Rm_{crit}
(o)	0	17.27	1.000	8.18	51.23
(ia)	10	17.30	0.983	8.15	48.20
(ib)	10	17.31	1.295	7.00	203.84
(ii)	1000	17.19	0.985	7.52	77.54

Table 1: Characteristics of the solutions of (2) with the forcing $F = 17.27$. E_K^P and E_K^T are the kinetic energies of the poloidal and toroidal parts of the inner motion. h is a specific helicity, $h = \langle \boldsymbol{u} \cdot \mathrm{curl}\ \boldsymbol{u} \rangle R / \langle \boldsymbol{u}^2 \rangle$ with $\langle \cdot \rangle$ denoting the average over one hemisphere. For the definition of Rm_{crit} see Section 4.

4. Dynamos

Let us now admit magnetic fields too. The influence of the fluid motion on them is characterized by the magnetic Reynolds number $Rm = UR/\eta$ with U as explained above. Clearly $Rm = Re\,Pm$, with $Pm = \nu/\eta$ being the magnetic Prandtl number.

We first restrict our attention on cases with weak magnetic fields only whose influence on the fluid motion is negligible. Let us introduce the Alfvén number $A = B/\sqrt{\mu\rho}U$ with B being the r.m.s. value of $\boldsymbol{B}$ inside the fluid body and U as explained above. In that sense we consider first the limit $A \to 0$.

Within this framework the question of stability or instability of the non–magnetic state of our system described by (1) and (2) is just the central question asked in the kinematic dynamo theory. In all cases listed in Table 1 the fluid motion is capable of dynamo action, that is, allows small magnetic fields to grow if only the electric conductivity of the fluid, or Rm, is sufficiently high. More precisely, this condition reads $Rm > Rm_{crit}$ with the values of Rm_{crit} given there. Note that they differ significantly between the two otherwise similar solutions (ia) and (ib) with $Ta = 10$. In all cases the magnetic fields that start to grow if Rm exceeds Rm_{crit} are of S1–type, that is, they are symmetric with respect to the equatorial plane and contain only the first harmonic with respect to the azimuth φ.

For all following considerations we assume $Ta \neq 0$ and put as an example $Pm = 4$. Then the non–magnetic state of our system is unstable in the case (ia) only but stable in all other cases.

For larger magnetic fields a genuine interaction between them and the fluid motion occurs. Under these conditions, that is, for non–vanishing Alfvén number A, two remarkable phenomena have been observed.

To describe the first one we start from a non–magnetic state of our system with the fluid flow of type (ia) and disturb it by a small magnetic field of S1–type. As indicated by the solid line in Fig. 2(a) initially the magnetic field grows. After reaching a sufficient strength it destroys the dynamo capability of the fluid flow and begins to decay. The evolution ends up with a zero magnetic field and a fluid flow of type (ib). That is, *the dynamo kills itself.* Fig. 3 gives some more details of the phenomenon considered. Here, the Alfvén number has initially the value $A = 0.009$, and its maximum during the evolution is $A = 0.08$.

To explain the second phenomenon we consider first a non–magnetic state of our system with a fluid flow of type (ii). Then small magnetic perturbations are bound to decay. With stronger perturbations, however, the flow can be deformed so that it gets capable of dynamo action. Then an evolution of the magnetic field as indicated by the solid line in Fig. 2(b) takes place which finally leads to a steady state with non–zero magnetic field. That is, *the magnetic field creates a dynamo.* The conditions under which this kind of evolution takes place are determined by the initial Alfvén number A and the initial geometrical structure of the magnetic field. Considering initial magnetic fields of S1–type only we found that the critical values of A are lower for poloidal than for toroidal field structures. Fig. 4 presents some details for two examples with different initial toroidal fields. In the example in which initially $A = 0.306$ a steady state is reached with $A = 0.260$ whereas for an initial value $A = 0.298$ no dynamo was created. In the former case motion and magnetic field are finally still symmetric with respect to the equatorial plane. In addition to the dominating axisymmetric part in the motion the even harmonics with respect to φ are present, and the magnetic field is enhanced by the higher odd harmonics.

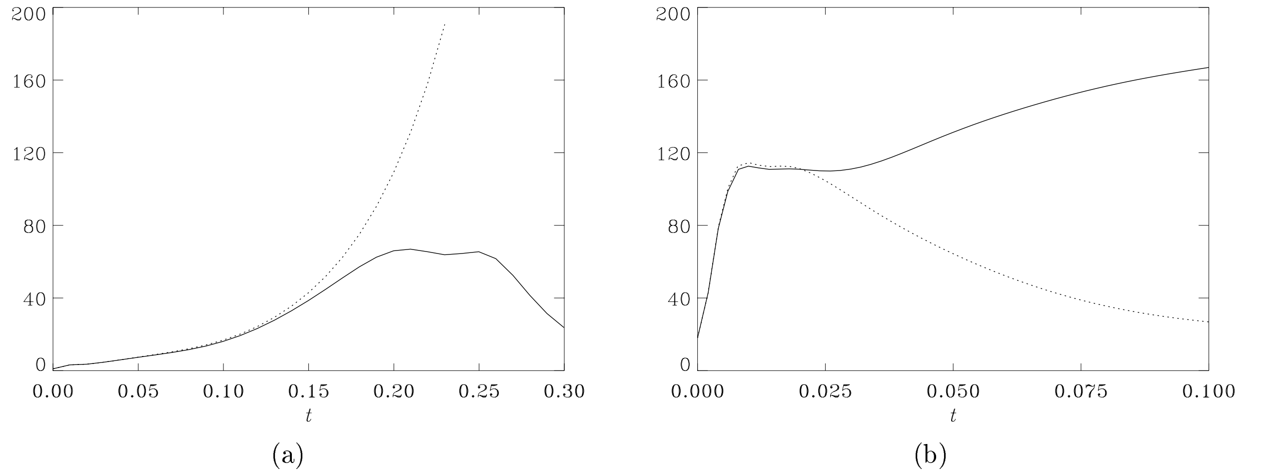

Fig. 2: The magnetic energy stored in all space, both fluid body and outer space, for the evolution starting with a fluid flow of type (ia) (left) or type (ii) (right) in units of $\rho\eta^2 R$ (solid lines). The time is measured in units of R^2/η. Dotted lines describe the evolution with the back–reaction of the magnetic field on the motions switched off.

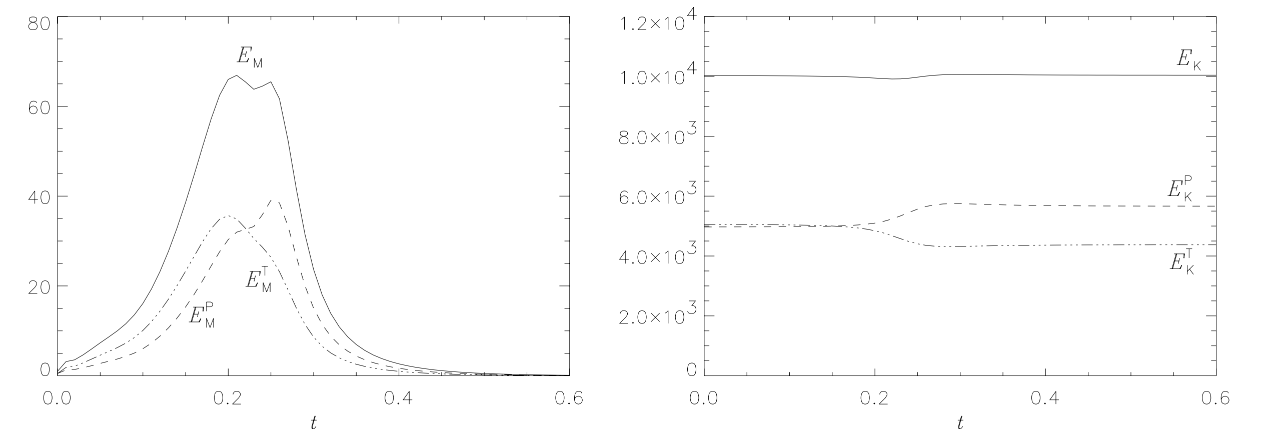

Fig. 3: The energy $E_{\rm M}$ of the magnetic field in all space, the energy $E_{\rm K}$ of the motion and the energies $E_{\rm M}^{\rm P}, E_{\rm M}^{\rm T}, E_{\rm K}^{\rm P}, E_{\rm K}^{\rm T}$ of the poloidal and toroidal parts of magnetic field and motion during the evolution starting from a nearly non–magnetic state with a fluid flow of type (ia). Units as in Fig. 2.

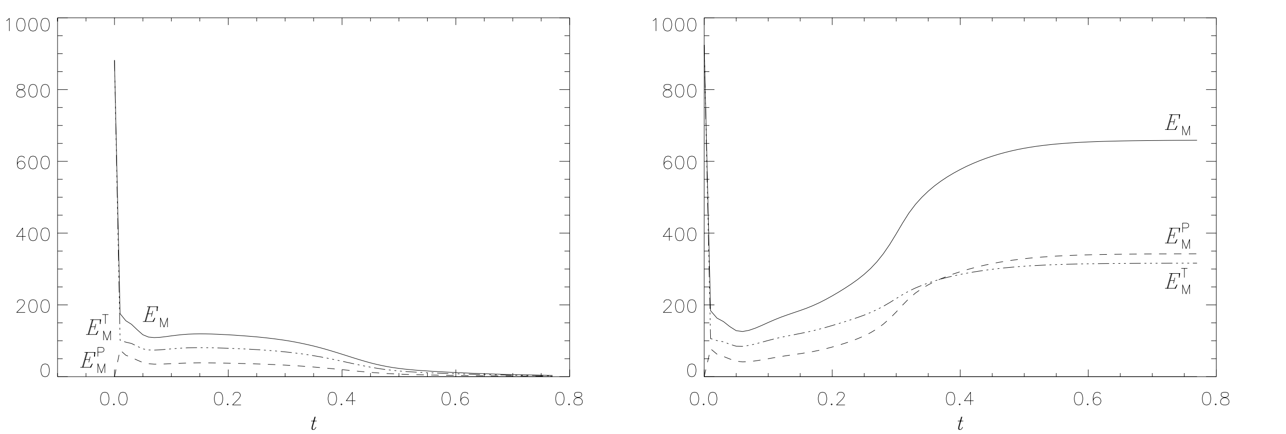

Fig. 4: The energies $E_{\rm M}$, $E_{\rm M}^{\rm P}$ and $E_{\rm M}^{\rm T}$ for evolutions starting from states with a fluid flow of type (ii) and toroidal fields with Alfvén numbers $A = 0.298$ (left) and $A = 0.306$ (right). For explanations see Fig. 3.

5. Conclusions

The first phenomenon described above provides us with an extreme example demonstrating that the behaviour of a system with genuine interaction of magnetic field and motion can drastically differ from the behaviour concluded from a kinematic model. Our finding implies a warning concerning simple parameterizations of the back–reaction of the magnetic field on the motion as used, e.g., in mean–field dynamo theory. The suicide of a dynamo as observed here is connected with the existence of more than one hydrodynamically stable states of the system, with and without stability with respect to magnetic perturbations. If the watershed between their basins of attraction is sufficiently low even a weak magnetic field can push the system from one basin to another, thus switching a dynamo on or off.

The second phenomenon is an example in which a sufficiently strong magnetic field organizes motions in favour of its maintenance. This requires, of course, the availability of kinetic energy, in our case delivered by the forcing. Note that dynamos of that kind work only in a regime with the full interaction of magnetic field and motion and have no counterparts on the kinematic level. As already mentioned another phenomenon of that kind has been discussed in the context of the Balbus–Hawley instability. It seems that such phenomena are not restricted to a very special situation but should be considered when discussing possibilities of the maintenance of magnetic fields.

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