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Stability of density-stratified viscous Taylor-Couette flows

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Abstract. The stability of density-stratified viscous Taylor-Couette flows is considered using the Boussinesq approximation but without any use of the short-wave approximation. The flows which are unstable after the Rayleigh criterion ($\hat{\mu} < \hat{\eta}^2$, with $\hat{\mu} = \Omega_{\rm out}/\Omega_{\rm in}$ and $\hat{\eta} = R_{\rm in}/R_{\rm out}$) now develop overstable axisymmetric Taylor vortices. For the considered wide-gap container we find the nonaxisymmetric modes as the most unstable ones. The nonaxisymmetric modes are unstable also *beyond the Rayleigh line*. For such modes the instability condition seems simply to be $\hat{\mu} < 1$ as stressed by Yavneh, McWilliams & Molemaker (2001). However, we never found unstable modes for too flat rotation laws fulfilling the condition $\hat{\mu} > \hat{\eta}$. The Reynolds numbers rapidly grow to very high values if this limit is approached (see Figs. 3 and 4). Also striking is that the marginal stability lines for the higher m do less and less enter the region beyond the Rayleigh line so that we might have to consider the stratorotational instability as a 'low-m instability'.
The applicability of these results to the stability problem of accretion disks with their strong stratification and fast rotation is shortly discussed.

Key words. Accretion, accretion disks, Hydrodynamics, Turbulence

1. Introduction

The flow pattern between concentric rotating cylinders with a stable axial density stratification was firstly studied by Thorpe (1966) who concluded that stable stratification stabilizes the flow. The further experimental and theoretical studies by Boubnov, Gledzer & Hopfinger (1995) and Caton, Janiaud & Hopfinger (2000) confirmed the stabilizing role of the density-stratification and showed that i) the critical Reynolds number depend on the buoyancy frequency (or Brunt-Väisälä frequency) of the fluid and ii) the stratification reduces the vertical extension of the Taylor vortices. The computational results of Hua, Le Gentil & Orlandi (1997) have indeed reproduced the experiment results.

The common feature of these studies is that the outer cylinder is at rest and flow is unstable after the Rayleigh condition for inviscid flow (which was extended to stratified fluids by Ooyama 1966), i.e.

$$\frac{\mathrm{d}}{\mathrm{d}R}(R^4\Omega^2) < 0 \tag{1}$$

where Ω is the angular velocity of the flow.

Recently, Molemaker, McWilliams & Yavneh (2001) and Yavneh, McWilliams & Molemaker (2001) found

$$\frac{\mathrm{d}\Omega^2}{\mathrm{d}R} < 0 \tag{2}$$

as the sufficient condition for (nonaxisymmetric) instability. The condition (2) is identical with the condition for magnetorotational instability of Taylor-Couette flow (Velikhov 1959). These results have been derived by a linear stability analysis for inviscid flow. The numerical results of Yavneh, McWilliams & Molemaker (2001) demonstrate the existence of the hydrodynamic instability also for finite viscosity.

An instability of density-stratified Taylor-Couette flow beyond the Rayleigh line $\hat{\mu} = \hat{\eta}^2$ for nonaxisymmetric disturbances have also been found experimentally (Withjack & Chen 1974). With their wide gap container ($\hat{\eta} = 0.2$) they found for $\hat{\mu} > 0$ the stability curve *crossing* the classical Rayleigh line. The observed instability was reported as *nonaxisymmetric*. The resulting experimental stability line, however, is very steep for positive $\hat{\mu}$ (see their Fig. 8) and does never cross the line $\hat{\mu} = \hat{\eta}$.

For real viscous flows there are very illustrative results by Yavneh, McWilliams & Molemaker (2001). In the present paper a more comprehensive study of such flows is given. The governing equations and the restrictions of the used Boussinesq approximation are discussed in Sect. 2 while the numerical results are presented in Sect. 3. Summary and final discussion are given in Sect. 4.

The existence of an instability in Taylor-Couette flows with a stable radial rotation law and with a stable z-

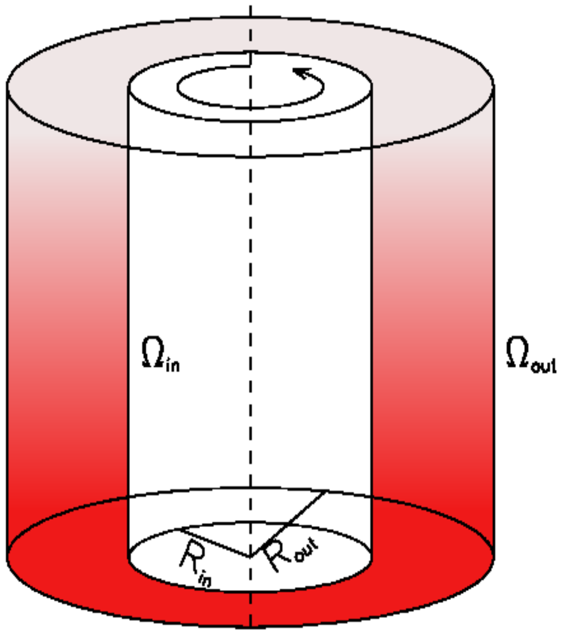

Fig. 1. The geometry of density-stratfied Taylor-Couette experiments.

stratification of the density is a surprise in the light of the Solberg-Høiland criterion (see Rüdiger, Arlt & Shalybkov 2002). The necessary condition for stability reads

$$\frac{\partial g_R}{\partial z} = \frac{\partial g_z}{\partial R}, \tag{3}$$

with $\boldsymbol{g}$ is the external force acceleration. Any *conservative* force is a particular solution of (3). Without external forces this relation is thus always fulfilled. If – as it is in accretion disks – the gravity balances the pressure and the centrifugal force, then Eq. (3) is automatically fulfilled. Note that after the Poincare theorem for rotating media with potential force and $\Omega = \Omega(R)$ both the density and the pressure can be written as functions of the generalized potential so that (3) is always fulfilled. Generally, the magnetic field is *not* conservative and can never fulfill the condition (3). This is the basic explanation for the existence of the magnetorotational instability (MRI) driven by (weak) magnetic fields.

Equation (3) has been derived by means of the short-wave approximation $m < R/\delta R$. For waves which are large in radial directions this condition might easily be fulfilled only for $m = 0$. It is, therefore, important to break the short-wave approximation in order to probe also the nonaxisymmetric modes. For Kepler flows (with uniform gravity acceleration) this has recently been done by Dubrulle et al. (2004). With Boussinesq approximation and direct numerical simulation for viscous flow they find all stratified flows with negative $\mathrm{d}\Omega/\mathrm{d}R$ unstable against nonaxisymmetric disturbances. According to their results a critical Froude number (as defined by Eq. 24) exists below which the flow is stable. Whether the Boussinesq approximation can be used for too small Froude numbers seems still to be an open question.

2. Equations and basic state

In cylindric coordinates (R, ϕ, z) the equations of incompressible stratified fluid with uniform dynamic viscosity, μ, are

$$\begin{aligned}
&\frac{\partial u_R}{\partial t} + (\boldsymbol{u}\nabla)u_R - \frac{u_\phi^2}{R} = \\
&\qquad -\frac{1}{\rho}\frac{\partial P}{\partial R} + \nu\left[\Delta u_R - \frac{2}{R^2}\frac{\partial u_\phi}{\partial \phi} - \frac{u_R}{R^2}\right], \\
&\frac{\partial u_\phi}{\partial t} + (\boldsymbol{u}\nabla)u_\phi + \frac{u_\phi u_R}{R} = \\
&\qquad -\frac{1}{\rho R}\frac{\partial P}{\partial \phi} + \nu\left[\Delta u_\phi + \frac{2}{R^2}\frac{\partial u_R}{\partial \phi} - \frac{u_\phi}{R^2}\right], \\
&\frac{\partial u_z}{\partial t} + (\boldsymbol{u}\nabla)u_z = -\frac{1}{\rho}\frac{\partial P}{\partial z} - g + \nu\Delta u_z, \\
&\frac{\partial u_R}{\partial R} + \frac{u_R}{R} + \frac{1}{R}\frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_z}{\partial z} = 0,
\end{aligned} \tag{4}$$

where

$$(\boldsymbol{u}\nabla)u_R = u_R\frac{\partial u_R}{\partial R} + \frac{u_\phi}{R}\frac{\partial u_R}{\partial \phi} + u_z\frac{\partial u_R}{\partial z} \tag{5}$$

and

$$\Delta u_R = \frac{\partial^2 u_R}{\partial R^2} + \frac{1}{R}\frac{\partial u_R}{\partial R} + \frac{1}{R^2}\frac{\partial^2 u_R}{\partial \phi^2} + \frac{\partial^2 u_R}{\partial z^2}. \tag{6}$$

ρ is the density, P is the pressure, g is the gravity, $\nu = \mu/\rho$ is the kinematic viscosity.[1] The equation which describes the evolution of the density fluctuation moving in the general density field is

$$\frac{\partial \rho}{\partial t} + (\boldsymbol{u}\nabla)\rho = 0. \tag{7}$$

We have to formulate the basic state with prescribed velocity profile $\boldsymbol{u} = (0, R\Omega(R), 0)$ and given density vertical stratification $\rho = \rho(z)$. The system (4) takes the form

$$\begin{aligned}
&\frac{u_\phi^2}{R} = \frac{1}{\rho}\frac{\partial P}{\partial R}, \qquad \frac{1}{\rho}\frac{\partial P}{\partial z} = -g, \\
&\frac{\partial^2 u_\phi}{\partial R^2} + \frac{1}{R}\frac{\partial u_\phi}{\partial R} - \frac{u_\phi}{R^2} = 0.
\end{aligned} \tag{8}$$

The last equation defines the angular velocity

$$\Omega = a + \frac{b}{R^2}, \tag{9}$$

where a and b are two constants related to the boundary values of the angular velocity, $\Omega_{\rm in}$, $\Omega_{\rm out}$, of the inner cylinder with radius, $R_{\rm in}$, and the outer cylinder with radius, $R_{\rm out}$. It follows

$$a = \Omega_{\rm in}\frac{\hat{\mu} - \hat{\eta}^2}{1 - \hat{\eta}^2}, \qquad b = \Omega_{\rm in}R_{\rm in}^2\frac{1 - \hat{\mu}}{1 - \hat{\eta}^2}, \tag{10}$$

[1] the density diffusion term in the mass conservation equation is neglected (see e.g. Caton, Janiaud & Hopfinger 2000)

with

$$\hat{\mu} = \frac{\Omega_{\rm out}}{\Omega_{\rm in}}, \qquad \hat{\eta} = \frac{R_{\rm in}}{R_{\rm out}}. \tag{11}$$

Differentiating the first equation of the system (8) by z and the second equation by R, subtracting each other and using the supposed profiles of density and angular velocity one gets

$$R\Omega^2 \frac{{\rm d}\rho}{{\rm d}z} = 0. \tag{12}$$

After this relation the density can depend only on the vertical coordinate z in the absence of rotation ($\Omega = 0$) and the angular velocity can only depend on radius in the absence of the vertical density stratification (${\rm d}\rho/{\rm d}z = 0$). The supposed profiles of the angular velocity, $\Omega = \Omega(R)$, and the density $\rho = \rho(z)$ are, therefore, not self-consistent. Thus, we must admit more general profile for the density $\rho = \rho(R, z)$ even though the initial stratification for the resting fluid is only vertical. In this case, the condition (12) takes the form

$$R\Omega^2 \frac{\partial \rho}{\partial z} + g\frac{\partial \rho}{\partial R} = 0. \tag{13}$$

The fluid transforms under the centrifugal force from the pure vertical stratification at the initial state to mixed (vertical and radial) stratification under the influence of the rotation strongly complicating the problem.

For real experiments the initial (without rotation) vertical stratification is small $|d{\rm log}\rho/d{\rm log}z \ll 1|$ as is the ratio of centrifugal acceleration to the vertical gravitation acceleration

$$\left|\frac{R^2\Omega}{g}\right| \ll 1, \tag{14}$$

so that after (13) the radial stratification is also small. Let us therefore consider the case of a weak stratification

$$\rho = \rho_0 + \rho_1(R, z), \qquad \rho_1 \ll \rho_0, \tag{15}$$

where ρ_0 is the uniform background density and (13) is fulfilled in zero-order. The perturbed state of the flow is described by

$$u_R, \qquad u_\phi + R\Omega(R), \qquad u_z,$$
$$P_0(R) + P_1(R, z) + P, \qquad \rho_0 + \rho_1(R, z) + \rho, \tag{16}$$

where $|P_1/P_0| \ll 1$ and u_R, u_ϕ, u_z, P and ρ are the perturbations. Linearizing the system (4) and selecting only the terms of the largest order we have

$$\frac{\partial u_R}{\partial t} + \Omega\frac{\partial u_R}{\partial \phi} - 2\Omega u_\phi = -\frac{1}{\rho_0}\frac{\partial P}{\partial R} + \frac{\rho}{\rho_0^2}\frac{\partial P_0}{\partial R} + $$
$$+\nu_0\left[\nabla u_R - \frac{2}{R^2}\frac{\partial u_\phi}{\partial \phi} - \frac{u_R}{R^2}\right],$$
$$\frac{\partial u_\phi}{\partial t} + \Omega\frac{\partial u_\phi}{\partial \phi} + \frac{1}{R}\frac{\partial}{\partial R}(R^2\Omega)u_R = -\frac{1}{\rho_0 R}\frac{\partial P}{\partial \phi} + $$
$$+\nu_0\left[\nabla u_\phi + \frac{2}{R^2}\frac{\partial u_R}{\partial \phi} - \frac{u_\phi}{R^2}\right],$$
$$\frac{\partial u_z}{\partial t} + \Omega\frac{\partial u_z}{\partial \phi} = -\frac{1}{\rho_0}\frac{\partial P}{\partial z} + \frac{\rho}{\rho_0^2}\frac{\partial P_0}{\partial z} + \nu_0\nabla u_z,$$
$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho_1}{\partial R}u_R + \Omega\frac{\partial \rho}{\partial \phi} + \frac{\partial \rho_1}{\partial z}u_z = 0,$$
$$\frac{\partial u_R}{\partial R} + \frac{u_R}{R} + \frac{1}{R}\frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_z}{\partial z} = 0 \tag{17}$$

with uniform $\nu_0 = \mu/\rho_0$. The first-order terms are left in the mass conservation equation due to vanishing of the zero-order terms.

Due to (14) we can neglect $\partial P_0/\partial R$ in the first equation and $\partial \rho_1/\partial R$ in the fourth arising from radial stratification (they will be $|R^2\Omega/g|$ times smaller than terms arising from the vertical stratifications) and the system takes exactly the Boussinesq form

$$\frac{\partial u_R}{\partial t} + \Omega\frac{\partial u_R}{\partial \phi} - 2\Omega u_\phi = -\frac{\partial}{\partial R}\left(\frac{P}{\rho_0}\right) + $$
$$+\nu_0\left[\nabla u_R - \frac{2}{R^2}\frac{\partial u_\phi}{\partial \phi} - \frac{u_R}{R^2}\right],$$
$$\frac{\partial u_\phi}{\partial t} + \Omega\frac{\partial u_\phi}{\partial \phi} + \frac{1}{R}\frac{\partial R^2\Omega}{\partial R}u_R = -\frac{1}{R}\frac{\partial}{\partial \phi}\left(\frac{P}{\rho_0}\right) + $$
$$+\nu_0\left[\nabla u_\phi + \frac{2}{R^2}\frac{\partial u_R}{\partial \phi} - \frac{u_\phi}{R^2}\right],$$
$$\frac{\partial u_z}{\partial t} + \Omega\frac{\partial u_z}{\partial \phi} = -\frac{\partial}{\partial z}\left(\frac{P}{\rho_0}\right) - g\frac{\rho}{\rho_0} + \nu_0\nabla u_z,$$
$$\frac{\partial}{\partial t}\left(\frac{\rho}{\rho_0}\right) + \Omega\frac{\partial}{\partial \phi}\left(\frac{\rho}{\rho_0}\right) - \frac{N^2}{g}u_z = 0,$$
$$\frac{\partial u_R}{\partial R} + \frac{u_R}{R} + \frac{1}{R}\frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_z}{\partial z} = 0, \tag{18}$$

where N is the vertical buoyancy frequency with

$$N^2 = -\frac{g}{\rho_0}\frac{\partial \rho_1}{\partial z}. \tag{19}$$

Suppose that the linear vertical density stratification $\partial \rho_1/\partial z$ = const and thus N^2 is a constant, too. Then the coefficients of the system (18) only depend on the radial coordinate and we can use a normal mode expansion of the solution $F = F(R){\rm exp}({\rm i}(m\phi + kz + \omega t))$ where F represents any of the disturbed quantities.

Let $D = R_{\rm out} - R_{\rm in}$ be the gap between the cylinders. We use $R_0 = (R_{\rm in}D)^{1/2}$ as the unit of length, the velocity $\Omega_{\rm in}R_0$ as the unit of the perturbed velocity, $\Omega_{\rm in}$ as the unit of ω, N and Ω. Using the same symbols for normalized quantities and redefining ρ as the dimensionless density $\rho g/\rho_0 R_0 \Omega_{\rm in}^2$ we finally find

$$\frac{\partial^2 u_R}{\partial R^2} + \frac{1}{R}\frac{\partial u_R}{\partial R} - \frac{u_R}{R^2} - \left(k^2 + \frac{m^2}{R^2}\right)u_R - 2{\rm i}\frac{m}{R}u_\phi - $$
$$-{\rm i}\ {\rm Re}(\omega + m\Omega)u_R + 2{\rm Re}\Omega u_\phi - {\rm Re}\frac{\partial P}{\partial R} = 0,$$
$$\frac{\partial^2 u_\phi}{\partial R^2} + \frac{1}{R}\frac{\partial u_\phi}{\partial R} - \frac{u_\phi}{R^2} - \left(k^2 + \frac{m^2}{R^2}\right)u_\phi + 2{\rm i}\frac{m}{R}u_R - $$
$$-{\rm i}\ {\rm Re}(\omega + m\Omega)u_\phi - {\rm i}\ {\rm Re}\frac{m}{R}P - \frac{\rm Re}{R}\frac{\partial}{\partial R}(R^2\Omega) = 0,$$

$$\frac{\partial^2 u_z}{\partial R^2} + \frac{1}{R}\frac{\partial u_z}{\partial R} - \left(k^2 + \frac{m^2}{R^2}\right) u_z -$$
$$-\mathrm{i}\ \mathrm{Re}(\omega + m\Omega)u_z - \mathrm{i}\ \mathrm{Re}\, kP - \mathrm{Re}\,\rho = 0,$$
$$\mathrm{i}(\omega + m\Omega)\rho - N^2 u_z = 0 \qquad (20)$$

and

$$\frac{\partial u_R}{\partial R} + \frac{u_R}{R} + \mathrm{i}\frac{m}{R}u_\phi + \mathrm{i}k u_z = 0 \qquad (21)$$

with the Reynolds number

$$\mathrm{Re} = \frac{\Omega_{\mathrm{in}} R_{\mathrm{in}} D}{\nu}. \qquad (22)$$

The standard no-slip boundary conditions used at the inner and outer cylinder, i.e.

$$u_R = u_\phi = u_z = 0, \qquad (23)$$

complete the classical eigenvalue problem. The same numerical method as in our previous papers about the Taylor-Couette problem (see e.g. Rüdiger & Shalybkov 2002) is used. Here we use a small negative imaginary part of ω to avoid problems with the corotation point $\omega = m\Omega$ for $m > 0$. Thus, the calculated critical Reynolds numbers are not for the marginally stable state but for slightly unstable state. To be sure that the calculated unstable state can be realized in experiments we checked the existence of the transition from stable to unstable state for several arbitrary points.

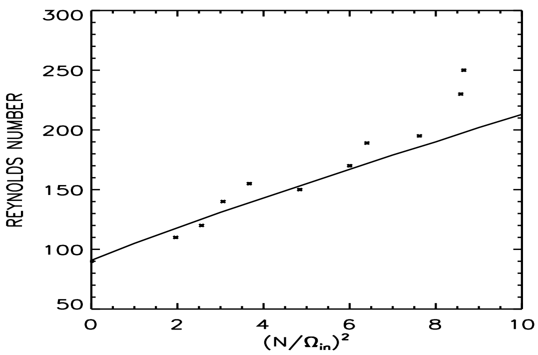

Fig. 2. The marginal stability line for axisymmetric disturbances ($m = 0$) for $\hat\eta = 0.78$, $\hat\mu = 0$. The dots represent the experimental data of Boubnov, Gledzer & Hopfinger (1995).

The code has been tested by computing for $m = 1$ the critical Reynolds number for the run 2 of Withjack & Chen (1974, their Tab. 1) with the experimental value 196.2 (with our normalizations) and our computed result 200.6 which we accepted to be in sufficiently good accordance.

3. Results

The imaginary parts of ω, $\Im(\omega)$, decrease with increasing Reynolds number. The Reynolds numbers above which the imaginary part of ω is smaller than some fixed value depend on the vertical wave number. They have a minimum at a certain wave number for fixed other parameters. This minimum value is called the critical Reynolds number.

In Fig. 2 we compare the calculated marginal stability line (i.e. $\Im(\omega) = 0$) for axisymmetric disturbances with experimental values by Boubnov, Gledzer & Hopfinger (1995). The agreement is rather good except the small values of the Froude number

$$\mathrm{Fr} = \frac{\Omega_{\mathrm{in}}}{N}. \qquad (24)$$

This disagreement may indicate the violation of the Boussinesq approximation. For Kepler disks we find $\mathrm{Fr} \simeq 0.5$ (Dubrulle et al. 2004). The unstratified fluids possess infinite Froude number.

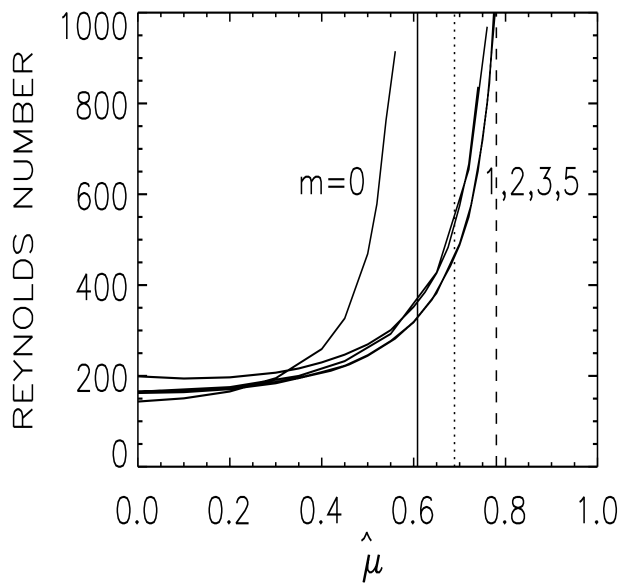

Fig. 3. The marginal stability line for $m = 0$ and critical Reynolds numbers for $m > 0$ for $\hat\eta = 0.78$ and $\mathrm{Fr} = 0.5$. The solid vertical line marks the $\hat\mu = \hat\eta^2$ limit, the dashed vertical line marks $\hat\mu = \hat\eta$ and the dotted vertical line marks $\hat\mu = \hat\eta^{1.5}$.

The dependence of the critical Reynolds numbers on $\hat\mu$ is given by Fig. 3 for a narrow gap and in Fig. 4 for a wide gap. The exact line of marginal stability is plotted only for $m = 0$. The axisymmetric disturbances are unstable only for $\hat\mu < \hat\eta^2$ in accordance to the Rayleigh condition (1). For $m > 0$ the slightly unstable lines with $\Im(\omega) = -10^{-3}$ are given.

The nonaxisymmetric disturbances are unstable also beyond the Rayleigh line (plotted as solid in the figures). The higher the m, however, the more the corresponding instability line approaches the Rayleigh line. Note, therefore, that the 'stratorotational instability' (SRI, Dubrulle et al. 2004) only produces low-m modes. This is an indication that indeed within the short-wave approximation (high-m) it does not exist (see Rüdiger, Arlt & Shalybkov 2002).

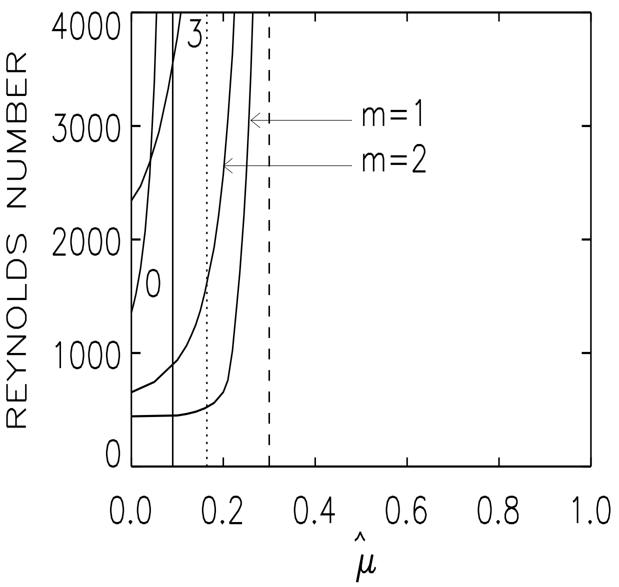

Fig. 4. The same as Fig. 3 but for a wide gap with $\hat{\eta} = 0.3$.

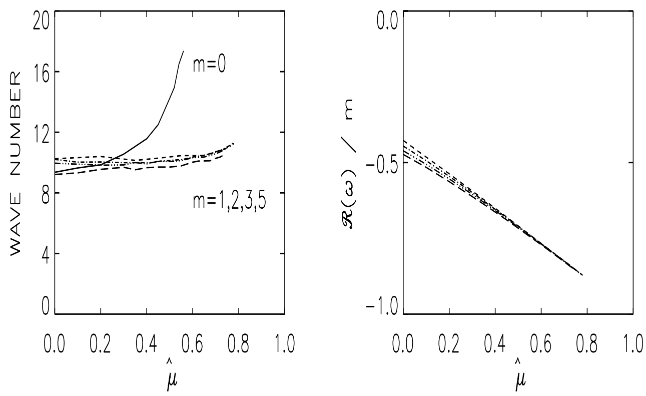

Fig. 5. The same as Fig. 3 but for vertical wave number (left) and the pattern speed $\Re(\omega)/m$ for $m > 0$ (right).

For nonstratified Taylor-Couette flows the nonaxisymmetric modes are only the most unstable disturbances for counter-rotating cylinders (e.g. Drazin & Reid 1981). The nonaxisymmetric instability of the stratified Taylor-Couette flow beyond the Rayleigh line ($\hat{\mu} > \hat{\eta}^2$) leads to the existence of some critical value, $\hat{\mu}_c$, beyond which the nonaxisymmetric disturbances are the most unstable. Our results show that $\hat{\mu}_c \sim 0.27$ and almost independent of the Froude number for $\hat{\eta} = 0.78$ (small gap) and $\hat{\mu}_c < 0$ for $\hat{\eta} = 0.3$ (wide gap). It is possible that for some $\hat{\eta}$ the nonaxisymmetric disturbances are the most unstable for all values of $\hat{\mu}$ corresponding to unstable flows.

There is another observation with the Figs. 3 and 4. Approaching the line $\hat{\mu} = \hat{\eta}$ the instability lines become more and more steep. We did not find any solution for $\hat{\mu} > \hat{\eta}$. If this is true one has to apply rather steep rotational profiles (not so steep as for nonstratified fluids but also not so weak as for the MRI) to find the modes of the linear SRI. This result seems to be of relevant for the discussion of the stability or instability of Kepler disks.

For the narrow gap ($\hat{\eta} = 0.78$) the critical Reynolds numbers of the nonaxisymmetric modes only slightly depend on m. The same is true for the critical vertical wave number and the pattern speed $\Re(\omega)/m$ (Fig. 5). The vertical wave number only weakly depends on $\hat{\mu}$ and the values $\Re(\omega)/m$ linearly run with $\hat{\mu}$. The situation is changed, however, for the wide gap ($\hat{\eta} = 0.3$). All parameters now strongly depend on both m and $\hat{\mu}$ (Fig. 6). The trend for the vertical wave numbers is opposite for $m = 3$ to those for $m = 1$ and $m = 2$.

The vertical wave numbers for both containers are of order 10 for $m = 1$. With our normalization the vertical extent of the Taylor vortices is given by

$$\frac{\delta z}{R_{\rm out}} = \frac{\pi}{k}\sqrt{\hat{\eta}(1-\hat{\eta})}. \quad (25)$$

With the mentioned value of k it is order of 0.1 for both the small-gap case and the wide-gap case. The cell becomes thus rather flat. For nonstratified TC-flows one finds $\delta z \simeq R_{\rm out} - R_{\rm in}$ while the cells under the influence of an axial magnetic field become more and more prolate. The stratification generally reduces the height of the Taylor vortices.

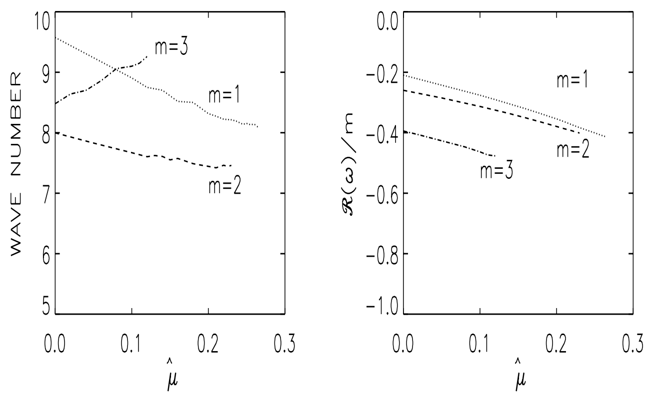

Fig. 6. The same as Fig. 5 but for $\hat{\eta} = 0.3$.

Unlike to the nonstratified Taylor-Couette flow the $\Re(\omega)$ is not zero for stratified Taylor-Couette even for axisymmetric disturbances ($m = 0$). The onset of instability is thus oscillatory ('overstability'). The question is whether a critical Froude number exists corresponding to the transition from stationary solutions to oscillating solutions? The answer is No. One cannot fulfill Eq. (20) for marginal stability ($\Im(\omega) = 0$) without a finite real part of ω for $N^2 \neq 0$. The axially stratified Taylor-Couette flow bifurcates from the purely azimuthal flow through a direct Hopf bifurcation to a wavy regime. Depending on the value of $\hat{\mu}$ and $\hat{\eta}$ this new regime can be either oscillating and axisymmetric or nonaxisymmetric and azimuthally drifting (see Figs. 3, 4). For both our containers the pattern speeds are negative for positive m. The drift of the spirals is thus always in the direction of the cylinder rotation.

Experiments have really demonstrated the oscillating onset of the axisymmetric instability (e.g. Caton, Janiaud & Hopfinger 2000). It would be interesting to design experiments with either rotating outer cylinder or wider gap to probe the bifurcation from the overstable oscillating axisymmetric flow pattern to the spiral nonaxisymmetric flow pattern.

As an example for the container with the narrow gap in Fig. 7 the velocity eigenfunctions are presented for $m = 1$ and for $\hat{\mu} = 0.7$ exceeding the value of $\hat{\eta}^2$. The functions are smooth enough and do not suggest that the instability must be explained as a boundary effect.

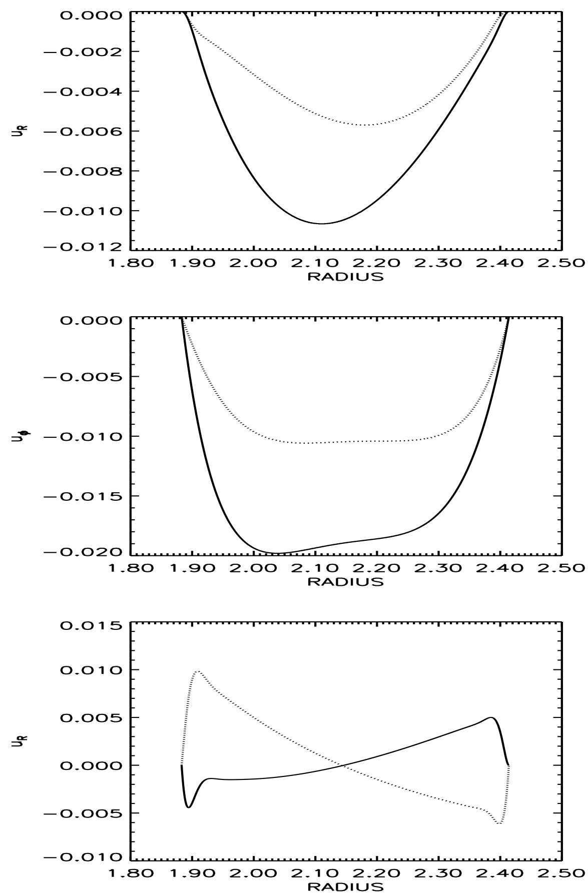

Fig. 7. The velocity eigenfunctions for $m = 1$, $\hat{\eta} = 0.78$, $\hat{\mu} = 0.7$, Fr = 0.5 with Reynolds number, vertical wave number and $\Re(\omega)$. The dotted lines are the real part and solid lines are the imaginary part.

The Reynolds number, the vertical wave number and the real part of ω for the transition from positive to negative imaginary part of ω are given in Fig. 8, i.e. for the transition from negative to positive growth rates. We find a continuous transition across the marginal stability line. It should thus be possible to realize the transition from stable to unstable flows in experiments. The vertical wave number and real part of ω are hardly influenced by the transition but, not surprisingly, the Reynolds numbers has a remarkably clear trend.

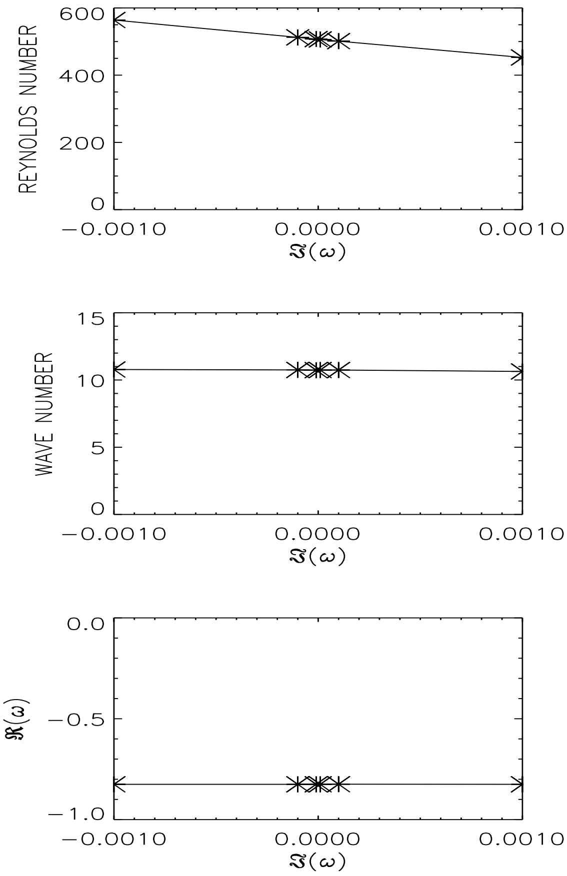

Fig. 8. The Reynolds number, vertical wave number and $\Re(\omega)$ in the transition from positive to negative $\Im(\omega)$ for $m = 1$, $\hat{\eta} = 0.78$, $\hat{\mu} = 0.7$, Fr = 0.5.

4. Discussion

It is shown that the Boussinesq approximation yields nonaxisymmetric disturbances with low m of the stratified Taylor-Couette flow as unstable even beyond the Rayleigh line $\hat{\mu} > \hat{\eta}^2$. Our results, however, also show that the critical Reynolds numbers are extremely increasing approaching the line $\hat{\mu} = \hat{\eta}$ so that as the condition for instability now the relation

$$\hat{\mu} < \hat{\eta} \tag{26}$$

seems to appear rather than $\hat{\mu} < 1$ according to Yavneh, McWilliams & Molemaker (2001) and Molemaker, McWilliams & Yavneh (2001). It is challenging to interprete the line $\hat{\mu} = \hat{\eta}$ with the (galactic) rotation profile u_ϕ =const in the same sense as to interprete the line $\hat{\mu} = \hat{\eta}^2$ with the rotation law for uniform specific angular momentum $R^2\Omega$ = const. Below we shall consider the line $\hat{\mu} = \hat{\eta}^{1.5}$ as concerning the Kepler flow.[2] Dubrulle et al. (2004) for their model of rotating plane Couette flow have found unstable solutions also beyond the line $\hat{\mu} = \hat{\eta}$ but also in these computations the rotation profile must be steeper than $R^{-2/3}$ (their Fig. 7).

The SRI also leads to the situation that nonaxisymmetric disturbances can be the most unstable modes not only for counterrotating cylinders but also for corotating cylinders. The characteristic values of $\hat{\mu}$ where the nonaxisymmetric disturbances are the most unstable ones strongly depend on the gap width. For $\hat{\eta} = 0.3$ all positive $\hat{\mu}$ are concerned (see Fig. 4). It cannot be excluded that the nonaxisymmetric disturbances are the most unstable ones for all $\hat{\mu}$ for $\hat{\eta}$ smaller than some critical value.

These results were obtained with the Boussinesq approximation so that two restrictions remain. The vertical density stratification should be weak enough and the rotation should be so slow that the centrifugal acceleration can be neglected in total. If one of these conditions is violated the Boussinesq approximation cannot be used and the situation becomes much more complicated. The disagreement between the calculated and the observed critical Reynolds numbers for small Fr (see Fig. 2) may already indicate the violation of the Boussinesq approximation for strong stratifications.

We have shown that for not too small negative $\mathrm{d}\Omega/\mathrm{d}R$ Taylor-Couette flows with vertical density stratification become unstable against nonaxisymmetric disturbance with $m = 1$ even if they are stable without density stratification. Kepler flows seem to be concerned by this phenomenon. In the Figs. 3, 4 the dotted lines represent the limit $\hat{\mu} = \hat{\eta}^{1.5}$ which might mimic the radial shear in Kepler disks. In both Figures we find a critical Reynolds number of only about 500 for the lowest ($m = 1$) mode. This is indeed a rather small number whose meaning, however, should not be overestimated. Approaching the line $\hat{\mu} = \hat{\eta}$ also these values become more and more large. Even more important is the finding that in magnetohydrodynamic Taylor-Couette experiments the MRI only needs *magnetic* Reynolds numbers of O(10). This leeds to hydrodynamic Reynolds numbers exceeding $\mathrm{O}(10^6)$ only for experiments with liquid metals in terrestrial laboratories. For hot plasma with magnetic Prandtl numbers of order 10 (Noguchi & Pariev 2003) the necessary hydrodynamic Reynolds number is also only O(1) or even smaller!

The existence of the SRI might be important for astrophysical applications. As suggested first by Richard & Zahn (1999) one should not forget (in particular for protoplanetary disks) to probe hydrodynamical instabilities as the source for the necessary turbulence in accretion disks. In partial confirmation of results of Dubrulle et al. (2004) we have here demonstrated that for the rotation laws of Taylor-Couette flows under the presence of vertical density gradients indeed linear hydrodynamic instabilities for low m exist. The properties of these modes are described above. Whether they are important for the accretion disk physics is still an open question. Note that the density stratification in accretion disks completely vanishes in the equatorial region and that the unstable modes discussed above would more or less lead to a nonaxisymmetric structure of the disk rather than to turbulence. The next step in this direction must include the numerical simulation for well-designed but simplified *global* accretion disk models.

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[2] note, however, the general difference of the Kepler rotation law $\Omega \propto R^{-1.5}$ and the Taylor-Couette rotation law (9)