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Tayler instability of toroidal magnetic fields in MHD Taylor-Couette flows

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The nonaxisymmetric Tayler instability (TI) of toroidal magnetic fields due to axial electric currents is studied for conducting incompressible fluids between two infinitely long corotating cylinders. For given Reynolds number of rotation the magnetic Prandtl number Pm of the liquid conductor and the ratio of the cylinder's rotation rates are the free parameters. It is shown that for resting cylinders the critical Hartmann number for instability does not depend on Pm hence the TI also exists in the limit $\mathrm{Pm} \to 0$. By rigid rotation the instability is suppressed where for $\mathrm{Pm} = 1$ the rotational quenching takes its maximum. Rotation laws with negative shear (i.e. $\mathrm{d}\Omega/\mathrm{d}R < 0$) strongly destabilize the toroidal field if the rotation is not too fast. In galaxies with their quadrupolar magnetic field geometry this effect could have drastic implications. For sufficiently high Reynolds numbers of rotation, however, the TI completely disappears. For the considered magnetic constellation the superrotation laws support the rotational stabilization. The angular momentum transport of the instability is anticorrelated with the shear so that an eddy viscosity can be defined which proves to be positive.
We have also shown the possibility of laboratory TI experiments with a wide-gap container filled with fluid metals like sodium or gallium. Even the effect of the rotational stabilization can be reproduced in the laboratory with electric currents of only a few kA.

1 Motivation

A known instability of toroidal fields is the current-driven ('kink-type') Tayler instability (TI) which is basically nonaxisymmetric (Tayler 1957; Vandakurov 1972; Tayler 1973; Acheson 1978). The toroidal field becomes unstable against nonaxisymmetric perturbations for a sufficiently large magnetic field amplitude depending on the radial profile of the field. A global rigid rotation of the system stabilizes the TI, i.e. much higher field amplitudes can be kept stable. For the rapidly rotating regime $\Omega^2 > \Omega_{\mathrm{A}}^2$ (with Ω_{A} being the Alfvén frequency of the toroidal field) the stability becomes complete, i.e. *all* possible modes in incompressible fluids of uniform density are stable (Pitts & Tayler 1985). We shall demonstrate in the present paper how this instability and its stabilization by rigid rotation can experimentally be realized with fluid conductors like sodium and gallium. There is so far no empirical or observational proof of the existence of the TI (Maeder & Meynet 2005).

Another important topic in this respect is the stability of rotation laws with $\mathrm{d}\Omega/\mathrm{d}R < 0$ ('subrotation'). It is known that they become centrifugally unstable in the hydrodynamic regime if they are steep enough to fulfill the Rayleigh criterion ($\mathrm{d}(R^2\Omega)/\mathrm{d}R < 0$). This linear instability is basically axisymmetric. However, for magnetized ideal fluids under rapid rotation, Acheson (1978) even finds instability of the nonaxisymmetric mode with $m = 1$ if the shear flow is 'superalfvénic', i.e.

$$-R\frac{\mathrm{d}\Omega^2}{\mathrm{d}R} > \Omega_{\mathrm{A}}^2. \tag{1}$$

Hence, a nonaxisymmetric MHD instability exists even for rather flat rotation laws if a weak toroidal magnetic field is present (despite of the rapid-rotation condition $\Omega^2 > \Omega_{\mathrm{A}}^2$). Of course, relation (1) has no meaning for vanishing magnetic fields. One can also say that Eq. (1) describes a destabilizing role of the differential rotation with $\mathrm{d}\Omega/\mathrm{d}R < 0$. The system of flow and field becomes unstable although the differential rotation alone would be stable and also the magnetic field alone would be stable.

It is, of course, important to know whether this result is modified for real fluids with finite values of viscosity and magnetic diffusivity. We shall show that indeed an extreme *destabilization* of toroidal magnetic fields by weak subrotation exists for modest rotation. More importantly, however, is the behavior of this nonaxisymmetric instability for strong differential rotation as the latter tends to destroy nonaxisymmetric magnetic patterns. We shall find that indeed the instability disappears for too fast rotation. The astrophysical consequences for the stability of toroidal magnetic fields in differentially rotating stellar radiative zones and in the fluid crust of high-spinning neutron stars might thus be very strong.

Important is also the existence of a nonaxisymmetric instability for flat subrotation laws even for current-free toroidal fields ($B_\phi \propto 1/R$) which is called Azimuthal

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MagnetoRotational Instability (AMRI, see Rüdiger et al. 2007a). It appears if the shear becomes superalfvénic, i.e. the magnetic Reynolds number exceeds the (high enough) Lundquist number of the toroidal field. For too high Reynolds numbers, however, also this effect disappears. Nonuniform rotation always tends to suppress any nonaxisymmetric magnetic mode. The same phenomenon can be observed for the nonaxisymmetric modes of TI.

Another new question arises about the role of 'superrotation' (i.e. $\mathrm{d}\Omega/\mathrm{d}R > 0$) which is rather stable in the hydrodynamic regime. One can expect that toroidal fields subject to superrotation may be stabilized. Then it should also be true that for solar low latitudes, where in the bulk of the convection zone the equatorial Ω increases outwards, the toroidal field is stabilized and can be amplified to much higher values than it would be possible for the opposite rotation law. Note that the sunspots with their rather high magnetic field strength appear in the same latitude as the superrotation does. In the present paper it is shown by use of a simplifying cylinder geometry that indeed for not too large magnetic Prandtl numbers superrotation stabilizes toroidal magnetic fields while subrotation strongly destabilizes toroidal magnetic fields in the case that the rotation is not too fast. The stabilization by superrotation, however, vanishes for large magnetic Prandtl number.

In the shearing box approximation Tagger, Pellat & Coroniti (1992) already considered nonaxisymmetric modes for vertical fields and also for azimuthal fields (Balbus & Hawley 1992). Except by these authors, the stability problem of a system of toroidal fields and differential rotation has been studied in cylindric geometry (Michael 1954; Chandrasekhar 1961; Howard & Gupta 1962; Chanmugam 1979; Knobloch 1992: Dubrulle & Knobloch 1993; Kumar, Coleman & Kley 1994; Pessah & Psaltis 2005; Shalybkov 2006) but in all studies only axisymmetric perturbations are considered. In ideal MHD also nonaxisymmetric modes have been studied for current-free toroidal fields (Ogilvie & Pringle 1996). Here as a continuation of papers by Rüdiger et al. (2007a,b) the attention is focused to the nonaxisymmetric perturbation modes with $m = 1$ for real fluids. In particular, the possible realizations of the instabilities as experiments in the (MHD-)laboratory are discussed.

2 The Taylor-Couette geometry

A Taylor-Couette container is considered confining a toroidal magnetic field of given amplitudes at the cylinders which rotate with different rotation rates Ω (see Fig. 1). The gap between the cylinders is considered small. For laboratory applications the case of a very wide gap is also considered. Formally, the inner radius is $100\,\hat{\eta}$ % of the outer radius. The extreme values of $\hat{\eta} = 0.05$ and $\hat{\eta} = 0.95$ are used in the present paper contrary to the calculations by Rüdiger et al. (2007a) for a medium gap of $\hat{\eta} = 0.5$. The fluid confined between the cylinders is assumed to be incompressible

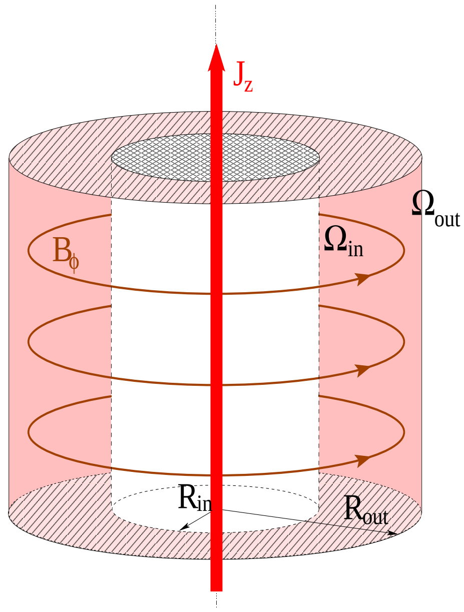

Fig. 1 (online colour at: www.an-journal.org) The conducting fluid resides between two concentric cylinders with radii R_{in} and R_{out} rotating with Ω_{in} and Ω_{out}. B_ϕ is the magnetic field due to axial currents inside the outer cylinder.

with uniform density and dissipative with the kinematic viscosity ν and the magnetic diffusivity η.

Derived from the conservation law of angular momentum the rotation law $\Omega(R)$ in the fluid is

$$\Omega(R) = a + \frac{b}{R^2}, \tag{2}$$

with

$$a = \frac{\mu_\Omega - \hat{\eta}^2}{1-\hat{\eta}^2}\Omega_{\mathrm{in}}, \qquad b = \frac{1-\mu_\Omega}{1-\hat{\eta}^2}R_{\mathrm{in}}^2\Omega_{\mathrm{in}}, \tag{3}$$

where

$$\hat{\eta} = \frac{R_{\mathrm{in}}}{R_{\mathrm{out}}}, \qquad \mu_\Omega = \frac{\Omega_{\mathrm{out}}}{\Omega_{\mathrm{in}}}. \tag{4}$$

Ω_{in} and Ω_{out} are the imposed rotation rates of the inner and outer cylinders with radii R_{in} and R_{out}. After the Rayleigh stability criterion the flow is hydrodynamically stable for $\mu_\Omega > \hat{\eta}^2$. We are only interested in hydrodynamically stable regimes so that $\mu_\Omega > \hat{\eta}^2$ must be fulfilled. Rotation laws with $\mathrm{d}\Omega/\mathrm{d}R > 0$ are described by $\mu_\Omega > 1$ and rotation laws with $\mathrm{d}\Omega/\mathrm{d}R < 0$ by $\mu_\Omega < 1$. $\mu_\Omega = 1$ gives the case of rigid rotation.

Also the magnetic profiles are restricted for real fluids. The solution of the stationary induction equation without inducing shear reads

$$B_\phi = AR + \frac{B}{R} \tag{5}$$

in cylinder geometry. A and B are the basic parameters; the term AR corresponds to uniform axial currents with $I = 2A$ everywhere within $R < R_{\mathrm{out}}$, and B/R corresponds to a uniform additional electric current only within $R < R_{\mathrm{in}}$. In the present paper we generally put

$$B = 0 \tag{6}$$

(see Pitts & Tayler 1985) with the consequence that the AMRI does not appear. The behavior of the toroidal field is thus only due to the TI for magnetic fields which are increasing outwards.

It is useful to define the quantity

$$\mu_B = \frac{B_{\rm out}}{B_{\rm in}}, \tag{7}$$

measuring the variation of B_ϕ across the gap. Vanishing B leads to $\mu_B = 1/\hat{\eta}$. For $\hat{\eta} \to 1$ this choice is so close to the current-free solution $\mu_B = \hat{\eta}$ that the field becomes unstable against perturbations with $m = 1$ only for very high Hartmann numbers[1].

In the following we fix $\mu_B = 1/\hat{\eta}$ but we shall vary the magnetic Prandtl number

$$\mathrm{Pm} = \frac{\nu}{\eta}, \tag{8}$$

and also the values of μ_Ω. If the results are to be applied to parts of the stellar convection zones the magnetic Prandtl number must be replaced by a value of order unity for the turbulent medium.

3 Equations and numerical model

The dimensionless MHD equations for incompressible fluids are

$$\mathrm{Re}\frac{\partial \boldsymbol{u}}{\partial t} + \mathrm{Re}(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla P + \Delta\boldsymbol{u} + \mathrm{Ha}^2\,\mathrm{rot}\,\boldsymbol{B}\times\boldsymbol{B},$$

$$\mathrm{Rm}\frac{\partial \boldsymbol{B}}{\partial t} = \Delta\boldsymbol{B} + \mathrm{Rm}\,\mathrm{rot}(\boldsymbol{u}\times\boldsymbol{B}), \tag{9}$$

with $\mathrm{div}\,\boldsymbol{u} = \mathrm{div}\,\boldsymbol{B} = 0$ and with the Hartmann number

$$\mathrm{Ha} = \frac{B_{\rm in}D}{\sqrt{\mu_0\rho\nu\eta}}. \tag{10}$$

Here $D = \sqrt{R_{\rm in}(R_{\rm out} - R_{\rm in})}$ is used as the unit of length, η/D as the unit of velocity and $B_{\rm in}$ as the unit of magnetic fields. Frequencies including the rotation Ω are normalized with the inner rotation rate $\Omega_{\rm in}$. The Reynolds number Re is defined as

$$\mathrm{Re} = \frac{\Omega_{\rm in}D^2}{\nu}, \tag{11}$$

and the magnetic Reynolds number is

$$\mathrm{Rm} = \frac{\Omega_{\rm in}\mathrm{D}^2}{\eta}. \tag{12}$$

Sometimes it appears here as useful to work with the 'mixed' Reynolds number

$$\mathrm{Rem} = \sqrt{\mathrm{Re}\cdot\mathrm{Rm}}, \tag{13}$$

which is symmetric in ν and η as is also the Hartmann number. For $\mathrm{Pm} = 1$ it is $\mathrm{Re} = \mathrm{Rm} = \mathrm{Rem}$. It is also useful to use the Lundquist number $\mathrm{S} = \sqrt{\mathrm{Pm}}\,\mathrm{Ha}$. The ratio of Rem and Ha,

$$\mathrm{Mm} = \frac{\mathrm{Rem}}{\mathrm{Ha}}, \tag{14}$$

is called the magnetic Mach number.

Applying the usual normal mode analysis, we look for solutions of the linearized equations of the form

$$F = F(R)\mathrm{exp}\big(\mathrm{i}(kz + m\phi + \omega t)\big). \tag{15}$$

Using Eq. (15), linearizing the Eq. (9) and representing the result as a system of first order equations, one finds

$$\begin{aligned}
&\frac{\mathrm{d}u_R}{\mathrm{d}R} + \frac{u_R}{R} + \mathrm{i}\frac{m}{R}u_\phi + \mathrm{i}ku_z = 0,\\
&\frac{\mathrm{d}P}{\mathrm{d}R} + \mathrm{i}\frac{m}{R}\phi_u + \mathrm{i}kZ + \left(k^2 + \frac{m^2}{R^2}\right)u_R +\\
&\qquad + \mathrm{iRe}(\omega + m\Omega)u_R - 2\Omega\mathrm{Re}u_\phi -\\
&\qquad - \mathrm{iHa}^2 mAb_R + 2\mathrm{Ha}^2 Ab_\phi = 0,\\
&\frac{\mathrm{d}\phi_u}{\mathrm{d}R} - \left(k^2 + \frac{m^2}{R^2}\right)u_\phi - \mathrm{iRe}(\omega + m\Omega)u_\phi +\\
&\qquad + 2\mathrm{i}\frac{m}{R^2}u_R - \frac{\mathrm{Re}}{R}\frac{\mathrm{d}}{\mathrm{d}R}\left(R^2\Omega\right)u_R +\\
&\qquad + 2\mathrm{Ha}^2 Ab_R + \mathrm{iHa}^2 mAb_\phi - \mathrm{i}\frac{m}{R}P = 0,\\
&\frac{\mathrm{d}Z}{\mathrm{d}R} + \frac{Z}{R} - \left(k^2 + \frac{m^2}{R^2}\right)u_z - \mathrm{iRe}(\omega + m\Omega)u_z -\\
&\qquad - \mathrm{i}kP + \mathrm{iHa}^2 mAb_z = 0,\\
&\frac{\mathrm{d}b_R}{\mathrm{d}R} + \frac{b_R}{R} + \mathrm{i}\frac{m}{R}b_\phi + \mathrm{i}kb_z = 0,\\
&\frac{\mathrm{d}b_z}{\mathrm{d}R} - \frac{\mathrm{i}}{k}\left(k^2 + \frac{m^2}{R^2}\right)b_R + \mathrm{PmRe}\frac{1}{k}(\omega + m\Omega)b_R +\\
&\qquad + \frac{1}{k}\frac{m}{R}\phi_B - \frac{1}{k}mAu_R = 0,\\
&\frac{\mathrm{d}\phi_B}{\mathrm{d}R} - \left(k^2 + \frac{m^2}{R^2}\right)b_\phi - \mathrm{iPmRe}(\omega + m\Omega)b_\phi +\\
&\quad + \mathrm{i}\frac{2m}{R^2}b_R - Ru_R + \mathrm{PmRe}R\frac{\mathrm{d}\Omega}{\mathrm{d}R}b_R + \mathrm{i}mAu_\phi = 0,
\end{aligned} \tag{16}$$

where ϕ_u, Z and ϕ_B are defined as

$$\phi_u = \frac{\mathrm{d}u_\phi}{\mathrm{d}R} + \frac{u_\phi}{R}, \quad Z = \frac{\mathrm{d}u_z}{\mathrm{d}R}, \quad \phi_B = \frac{\mathrm{d}b_\phi}{\mathrm{d}R} + \frac{b_\phi}{R}, \tag{17}$$

and $A = 1/R_{\rm in}$ ($R_{\rm in}$ in units of D).

An appropriate set of ten boundary conditions is needed to solve the system (16). For the velocity the boundary conditions are always no-slip, i.e.

$$u_R = u_\phi = u_z = 0. \tag{18}$$

For conducting walls the radial component of the field and the tangential components of the current must vanish yielding

$$\mathrm{d}b_\phi/\mathrm{d}R + b_\phi/R = b_R = 0. \tag{19}$$

These boundary conditions are applied at both $R_{\rm in}$ and $R_{\rm out}$. The wave number is varied as long as for given Hartmann number the Reynolds number takes its minimum. The procedure is already described in detail by Shalybkov, Rüdiger & Schultz (2002). One immediately finds that the sign of the real wave number k is free so that with the solution for k also another one with $-k$ exists with the same

[1] $\mathrm{Ha}_{\rm crit} = \infty$ for current-free magnetic fields without rotation

eigenfrequency. For containers bounded in z standing waves can thus develop.

The necessary and sufficient condition for the stability of toroidal fields in ideal Taylor-Couette flows against axisymmetric perturbations is by Michael (1954) and reads

$$\frac{1}{R^3}\frac{\mathrm{d}}{\mathrm{d}R}(R^2\Omega)^2 - \frac{R}{\mu_0\rho}\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{B_\phi}{R}\right)^2 > 0\,. \tag{20}$$

Fields which are not steeper than $B_\phi \propto R$ are thus always stable against $m = 0$ perturbations if the rotation law is also stable. Tayler (1973) found the necessary and sufficient condition

$$\frac{\mathrm{d}}{\mathrm{d}R}\left(RB_\phi^2\right) < 0 \tag{21}$$

for stability of an ideal nonrotating fluid against nonaxisymmetric disturbances. Our field profile $B_\phi = AR$ is thus stable against $m = 0$ and unstable for sufficiently large field amplitudes, i.e. for $\mathrm{Ha} > \mathrm{Ha}_{\mathrm{crit}}$. The same is true for the nearly uniform fields with $\mu_B \simeq 1$. A general criterion for rotating flows does not exist.

The system (16) is used to demonstrate that for resting cylinders the $\mathrm{Ha}_{\mathrm{crit}}$ does not depend on the magnetic Prandtl number Pm. To this end in the equations all terms resulting from the rotational influence are dropped. The frequency ω is replaced by ω/Re. Furthermore, the flow components u_ϕ and u_z are replaced by $-\mathrm{i}u_\phi$ and $-\mathrm{i}u_z$ and the field component b_R is replaced by $\mathrm{i}b_R$. It results

$$\begin{aligned}
&\frac{\mathrm{d}u_R}{\mathrm{d}R} + \frac{u_R}{R} + \frac{m}{R}u_\phi + ku_z = 0,\\
&\frac{\mathrm{d}P}{\mathrm{d}R} + \frac{m}{R}\phi_u + kZ + \left(k^2 + \frac{m^2}{R^2}\right)u_R +\\
&\qquad + \mathrm{i}\omega u_R + \frac{\mathrm{Ha}^2 m}{R_{\mathrm{in}}}b_R + \frac{2\mathrm{Ha}^2}{R_{\mathrm{in}}}b_\phi = 0,\\
&\frac{\mathrm{d}\phi_u}{\mathrm{d}R} - \left(k^2 + \frac{m^2}{R^2}\right)u_\phi - \mathrm{i}\omega u_\phi - 2\frac{m}{R^2}u_R -\\
&\qquad - \frac{2\mathrm{Ha}^2}{R_{\mathrm{in}}}b_R - \frac{\mathrm{Ha}^2 m}{R_{\mathrm{in}}}b_\phi + \frac{m}{R}P = 0,\\
&\frac{\mathrm{d}Z}{\mathrm{d}R} + \frac{Z}{R} - \left(k^2 + \frac{m^2}{R^2}\right)u_z - \mathrm{i}\omega u_z +\\
&\qquad + kP - \frac{\mathrm{Ha}^2 m}{R_{\mathrm{in}}}b_z = 0,\\
&\frac{\mathrm{d}b_R}{\mathrm{d}R} + \frac{b_R}{R} + \frac{m}{R}b_\phi + kb_z = 0,\\
&\frac{\mathrm{d}b_z}{\mathrm{d}R} + \frac{1}{k}\left(k^2 + \frac{m^2}{R^2}\right)b_R + \mathrm{Pm}\frac{\mathrm{i}}{k}\omega b_R +\\
&\qquad + \frac{m}{kR}\phi_B - \frac{m}{kR_{\mathrm{in}}}u_R = 0,\\
&\frac{\mathrm{d}\phi_B}{\mathrm{d}R} - \left(k^2 + \frac{m^2}{R^2}\right)b_\phi - \mathrm{iPm}\omega b_\phi -\\
&\qquad - \frac{2m}{R^2}b_R - Ru_R + \frac{m}{R_{\mathrm{in}}}u_\phi = 0,
\end{aligned} \tag{22}$$

Note that the magnetic Prandtl number only survives together with $\mathrm{i}\omega$ which is purely imaginary for marginal instability. Hence, in the real part of the system (22) the frequency terms including the Pm do not appear. The only free parameter in the real part of the system (22), therefore, is the critical Hartmann number which results thus as equal for all Pm. Shalybkov (2006) has given a similar result for $m = 0$.

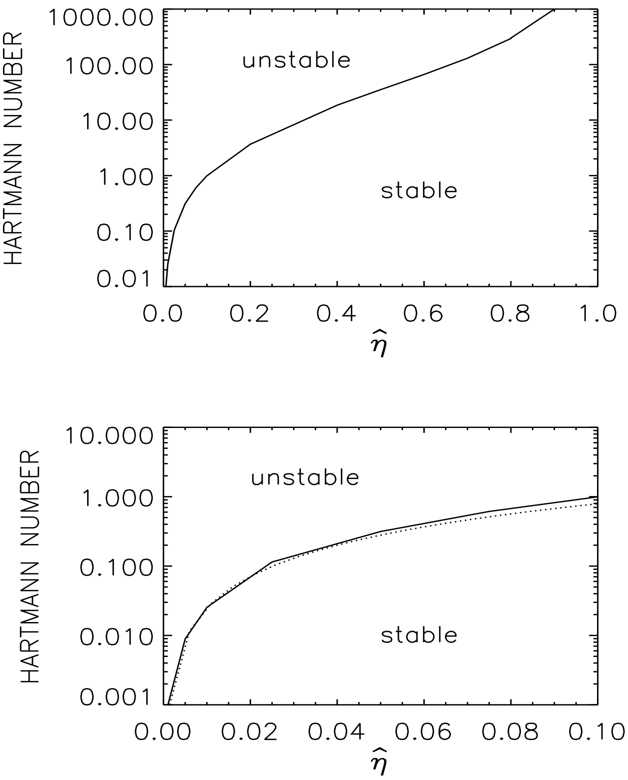

Fig. 2 The critical Hartmann number for $\Omega = 0$ for various container gaps ($\hat{\eta}$) between conducting cylinders. *Bottom*: very wide gaps ($\hat{\eta} \to 0$). The dotted line represents the algebraic expression $25\,\hat{\eta}^{1.5}$. The numbers are valid for all magnetic Prandtl numbers, see text.

The Hartmann numbers $\mathrm{Ha}_{\mathrm{crit}}$ for various $\hat{\eta}$ are given in Fig. 2. They vary over many orders of magnitude and become very small for wide gaps ($\hat{\eta} \to 0$). For small $\hat{\eta}$ the critical Hartmann number vanishes like

$$\mathrm{Ha}_{\mathrm{crit}} \propto \hat{\eta}^{1.5} \tag{23}$$

with a factor of about 25. Our calculation with the smallest R_{in} concern $\hat{\eta} = 0.001$ and lead to $\mathrm{Ha} = 0.00075$.

In the present paper the gap between the cylinders is assumed as narrow ($\hat{\eta} = 0.95$) and in another computation as wide ($\hat{\eta} = 0.05$) so that the unit of distances, D, is the same in both cases for fixed outer radius. The narrow gap model serves to astrophysical discussions (solar tachocline, neutron star crust, supergranulation layer) while the wide gap results are needed to prepare future laboratory experiments. We have shown with similar models that indeed wide gaps are much more suitable for TI experiments with liquid metals like sodium or gallium than narrow gaps (Rüdiger et al. 2007b).

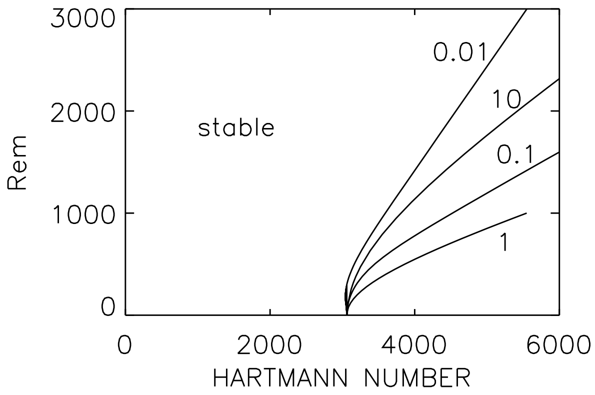

Fig. 3 The suppression of the TI by rigid rotation. The curves are marked with their magnetic Prandtl number Pm. The rotational suppression of TI is weaker for $\mathrm{Pm} \neq 1$ than for $\mathrm{Pm} = 1$. $\hat{\eta} = 0.95$, narrow gap.

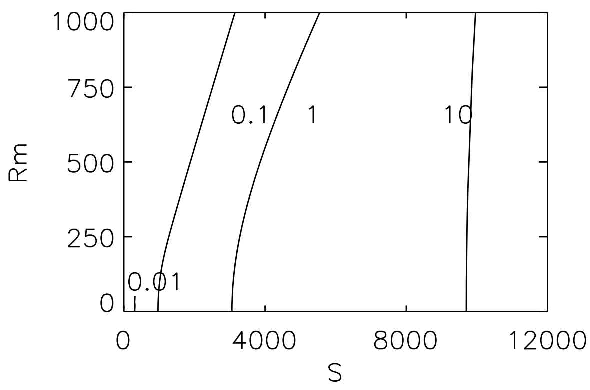

Fig. 4 The same as in Fig. 3 but for fixed magnetic diffusivity η and free molecular viscosity. High viscosity stabilizes the magnetic fields and reduces the rotational influence.

4 Narrow gap

For $\hat{\eta} = 0.95$ it is $\mu_B = 1.05$. The critical Hartmann number for $\mathrm{Re} = 0$ is 3061 for all Pm (see Fig. 2). The Rayleigh limit for centrifugal instability is $\mu_\Omega = \hat{\eta}^2 = 0.9025$.

4.1 Rigid rotation

We start with the simplest case of the interaction of toroidal field and global rotation, i.e. with the marginal instability for rigid rotation ($\mu_\Omega = 1$) and for various Pm.

The results of the calculations are shown in Fig. 3 in the plane of Rem and Ha. As we already know the critical Hartmann number Ha for $\mathrm{Rem} = 0$ does not depend on the magnetic Prandtl number. The new result is that the critical Ha always grows for growing rotation rate. This is the *stabilizing* action of rotation. In the representation of Fig. 3 (where the parameters on both axes are symmetric in ν and η) the rotational stabilization is strongest for $\mathrm{Pm} = 1$ but it becomes weaker for $\mathrm{Pm} \neq 1$. For $\mathrm{Rem} \lesssim 500$ the differences of the critical magnetic fields for Pm varying over three orders of magnitude are surprisingly small.

For higher values of Rem the curves seem to represent a *linear* relation between Rem and Ha. Note that in Fig. 3 the magnetic Mach number

$$\mathrm{Mm} = \frac{\Omega D}{B_{\mathrm{in}}/\sqrt{\mu_0 \rho}} \tag{24}$$

is the smallest for $\mathrm{Pm} = 1$. If this behavior remained true also for much higher Ha then the consequences can be strong: In turbulent fluids and/or in simulations with $\mathrm{Pm} = 1$ rather strong magnetic fields remain stable which for the real $\mathrm{Pm} \neq 1$ are already unstable. For small Pm (stellar radiative interior) and high Pm (galaxies, neutron stars) the magnetic instability is much more efficient and already works for much smaller magnetic field strengths.

For neutron stars the ratio (24) yields

$$\mathrm{Mm} \simeq \frac{3 \times 10^{14}\ \mathrm{Gauss}}{B_\phi}, \tag{25}$$

so that $B_\phi \simeq 3 \times 10^{14}$ Gauss is the critical value for the toroidal field. We have here used the numerical values $\Omega = 100\ \mathrm{s}^{-1}$, $\rho \simeq 10^{13}$ g/cm^3 and $D \simeq 3 \times 10^5$ cm.

The same data are plotted in Fig. 4 in the plane of Rm and S. Again the curves are marked with their magnetic Prandtl numbers. One can read this plot in the sense that rotation, magnetic field and magnetic diffusivity η are given and the viscosity (in units of η) is varied. For high viscosity the fields must be much stronger to become unstable. One finds a distinct stabilizing influence of the viscosity. The rotational influence, however, is weaker for high viscosity.

The results are applied to the bottom of the convection zone. The theory of the advection-dominated dynamo requires a small value of 10^{11} cm^2/s for the eddy diffusivity η while the eddy viscosity ν should be larger for the explanation of the differential rotation pattern so that $\mathrm{Pm} \geq 10$. With $\Omega \simeq 2 \times 10^{-6}\ \mathrm{s}^{-1}$ and $D \simeq 10^{10}$ cm we find $\mathrm{Rm} \simeq 2000$. Figure 4 yields $\mathrm{S} \leq 10^4$ for stability. With $\rho \simeq 0.1$ g/cm^3 the upper limit for stable toroidal fields becomes 100 kGauss. Stronger fields will not be possible.

4.2 Nonuniform rotation

The form of the rotation law is now changed for various magnetic Prandtl numbers. Rotation laws with negative shear (here $\mu_\Omega = 0.5$ and $\mu_\Omega = 0.92$) and with negative shear (here $\mu_\Omega = 1.07$) are investigated. The rotation law with $\mu_\Omega = 0.5$ is centrifugally unstable without magnetic field. It is given only for demonstration. The remaining rotation laws are stable in the hydrodynamic regime. One finds the numerical results for marginal stability of the $m = 1$ mode in Fig. 5. Superrotation stabilizes the field more than solid-body rotation. For subrotation the behavior is opposite. While for rigid rotation and superrotation the critical Hartmann numbers grow for growing Reynolds number Rem, for subrotation the Ha become smaller so that finally the shear becomes superalfvénic. This effect is in accordance with the Acheson relation (1).

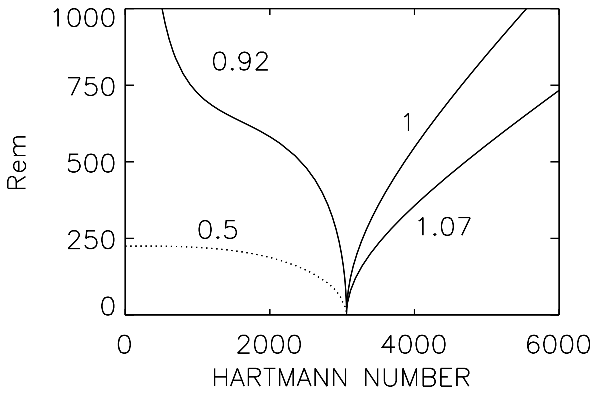

Fig. 5 Reynolds number vs. Hartmann number of the TI for subrotation ($\mu_\Omega = 0.5$, $\mu_\Omega = 0.92$), rigid rotation ($\mu_\Omega = 1$) and superrotation ($\mu_\Omega = 1.07$). Note the rotational quenching for $\mu_\Omega = 1$, the strong destabilization by subrotation and the stabilization by superrotation. Conducting cylinder walls, $\hat{\eta} = 0.95$, $\mathrm{Pm} = 1$.

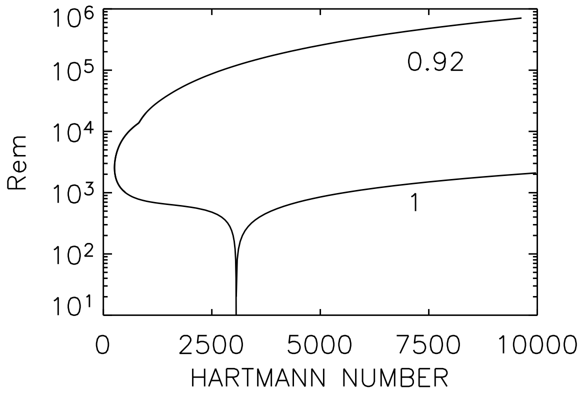

Fig. 6 The same as in Fig. 5 for $\mu_\Omega = 0.92$ and for solid-body rotation ($\mu_\Omega = 1$) but for much higher Reynolds numbers. The destabilization by differential rotation only holds for medium Reynolds numbers. For faster rotation the rotational stabilization dominates. $\mathrm{Pm} = 1$.

This finding, however, cannot be the final answer. Very intensive differential rotation always tends to suppress non-axisymmetric magnetic patterns. We therefore expect that for higher Reynolds numbers the Eq. (1) looses its meaning. In Fig. 6 the marginal-instability curve for $\mu_\Omega = 0.92$ is thus followed to higher values of the Reynolds number. The result of the calculations is that also for subrotation the basic rotation finally suppresses the instability if $\mathrm{Rem} \gtrsim 4000$. However, the rigid rotation seems to be much more effective to stabilize the TI. The critical magnetic amplitudes for instabilities are much higher for rigid rotation than for differential rotation. There is, surprisingly, no extra stabilization effect of the differential rotation in the considered Taylor-Couette flow.

Up to this value the differential rotation acts *destabilizing*. Applications of such results to astrophysical objects are possible but only for magnetic geometries symmetric with respect to the equator as existing in galaxies. Toroidal field belts with different signs in both hemispheres behave different (see Rüdiger & Kitchatinov 2010). In regions of convection zones with $\mathrm{d}\Omega/\mathrm{d}R < 0$ much weaker toroidal fields become unstable than for rigid rotation. This finding should have strong implications for the electrodynamics of rotating stars with quadrupolar magnetic field geometry. Young stars typically rotate ten times faster than old stars such as the Sun. For the supergranulation part of the solar convection zone the turbulent magnetic Reynolds number does not exceed (say) 300. Due to their faster rotation for younger solar-type stars it can easily reach the value of 3000. For all Pm and a Reynolds number of order 3000 the field becomes unstable already for very small Ha of order 100. Hence, the maximum stable toroidal fields in this zone is much weaker for fast rotating stars than for the slow-rotating Sun.

In Fig. 5 also the (dashed) curve for $\mu_\Omega = 0.5$ is given for comparison. Such a rotation law is unstable without magnetic field for $m \geq 0$. The magnetic field additionally destabilizes the rotation law so that for $\mathrm{Ha} = 3061$ the TI works even without any rotation.

The differences of the results for subrotation and for superrotation only appear for faster rotation but it is still unclear how strong the magnetic fields must be. The calculations for nonuniform rotation laws must be extended to smaller and higher magnetic Prandtl numbers. Figure 7 gives the results for $\mathrm{Pm} = 0.1$ and $\mathrm{Pm} = 10$. They can be best written with the characteristic numbers Rem and Ha because in this formulation the differences for the small and the large magnetic Prandtl number are smallest. For small magnetic Prandtl number the stabilizing influence of superrotation is stronger than for high magnetic Prandtl numbers. Note that for $\mathrm{Pm} = 10$ the stabilization of the magnetic field by superrotation is even smaller than that of rigid rotation. It is not clear whether for even larger magnetic Prandtl numbers rotation laws with positive shear become able to destabilize the magnetic fields.

Written with Rem and Ha one finds for given rotation law with $\mu_\Omega \simeq 1$ that $\mathrm{Pm} = 1$ yields the most effective stabilization of the toroidal magnetic field. For $\mathrm{Pm} \neq 1$ the stabilization is stronger for small Pm than for large Pm. The differences, however, are small. For subrotation ($\mu_\Omega < 1$) the very effective rotational destabilization of the magnetic field hardly depends on the magnetic Prandtl number.

In summary, we have shown for the container with the narrow gap that without rotation the critical Hartmann number is $\mathrm{Ha} = 3061$ independent of the magnetic Prandtl number. This value is increased for solid-body rotation and for superrotation but it is drastically reduced to about 100 for subrotation with $\mathrm{Rem} \geq 1000$. There is no strong dependence of these characteristic numbers on the magnetic Prandtl number. However, for too fast rotation ($\mathrm{Rem} > 3500$) the destabilization changes to rotational stabilization (see Fig. 6). On the other hand, the role of superrotation to stabilize the magnetic field changes to destabilization for too high magnetic Prandtl number.

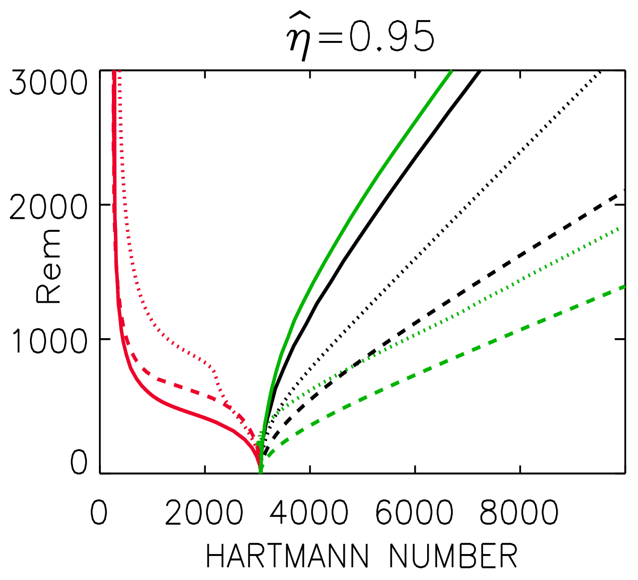

Fig. 7 (online colour at: www.an-journal.org) The same as in Fig. 5 but for $\mathrm{Pm} = 0.1$ (dotted), $\mathrm{Pm} = 1$ (dashed) and $\mathrm{Pm} = 10$ (solid). The curves for negative shear ($\mu_\Omega = 0.92$, subrotation) are red and the curves for positive shear ($\mu_\Omega = 1.07$, superrotation) are green. For large Pm superrotation yields less stability than rigid rotation.

4.3 The angular momentum transport

The solutions of the linear equations are free to an arbitrary real parameter of any sign. We do not know, therefore, the sign of the flows and the fields. However, for quadratic expressions such as the correlation tensor or the electromotive force one can find the signs as all the solutions are multiplied with one and the same parameter.

Let us apply this idea to the angular momentum transport

$$T_R = \langle u'_R u'_\phi - \frac{1}{\mu_0\,\rho} B'_R B'_\phi \rangle. \tag{26}$$

The average procedure consists of an integration over the azimuth ϕ. The question we shall answer is whether T_R and $\mathrm{d}\Omega/\mathrm{d}R$ are anticorrelated. If this is true then the angular momentum flows towards the minimum of the angular velocity, and one can introduce an eddy viscosity ν_T in accordance to

$$T_R = -\nu_\mathrm{T} R \frac{\mathrm{d}\Omega}{\mathrm{d}R} \tag{27}$$

with positive ν_T.

After normalization the expression (26) reads

$$T_R \simeq \langle u'_R u'_\phi \rangle - \mathrm{Ha}^2 \mathrm{Pm} \langle B'_R B'_\phi \rangle. \tag{28}$$

In Fig. 8 the angular momentum (26) normalized with $\sqrt{\langle u'^2_R u'^2_\phi \rangle}$ is given. Without magnetic fields its absolute value must be smaller than unity. The Maxwell stress, however, may produce higher values. We see from Fig. 8 that in the linear theory the magnetic contribution is surprisingly small.

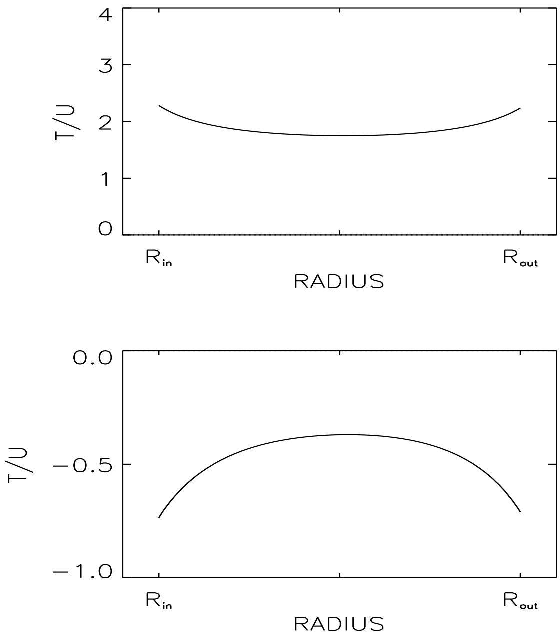

Fig. 8 The angular momentum transport (26) for subrotation ($\mu_\Omega = 0.92$, top) and superrotation ($\mu_\Omega = 1.07$, bottom). $\mathrm{Re} = 500$, $\mathrm{Pm} = 1$. The torque is positive for subrotation and negative for superrotation. The values are normalized with the turbulence intensity $U = \sqrt{\langle u'^2_R \rangle \langle u'^2_\phi \rangle}$. Due to the Maxwell stress they can exceed unity.

For simplicity we only work here with $\mathrm{Pm} = 1$. The angular momentum transport vanishes for rigid rotation and it is indeed anticorrelated with $\nabla\Omega$. The diffusion approximation (27) is thus possible. Its magnetic part is (only) of the same order than its kinetic part.

5 Wide gap

For $\hat{\eta} = 0.05$ it is $\mu_B = 1/\hat{\eta} = 20$. The critical Hartmann number for $\mathrm{Re} = 0$ is 0.31 for all Pm (see Fig. 2). The Rayleigh limit for centrifugal instability is $\mu_\Omega = 0.0025$.

5.1 Electric currents

The technical possibilities are now discussed to realize such a small Hartmann number in the laboratory working with liquid metals. Let I_axis be the axial current inside the inner cylinder and I_fluid the axial current between the inner and the outer cylinder. The assumption $B = 0$ in Eq. (5) provides

$$\frac{I_\mathrm{fluid}}{R^2_\mathrm{out} - R^2_\mathrm{in}} = \frac{I_\mathrm{axis}}{R^2_\mathrm{in}} \tag{29}$$

as the current density is homogeneous. Hence,

$$\frac{I_\mathrm{fluid}}{I_\mathrm{axis}} = \frac{1-\hat{\eta}^2}{\hat{\eta}^2} = 399 \tag{30}$$

for $\hat{\eta} = 0.05$. It is also clear that

$$B_\mathrm{in} = \frac{I_\mathrm{axis}}{5 R_\mathrm{in}}, \tag{31}$$

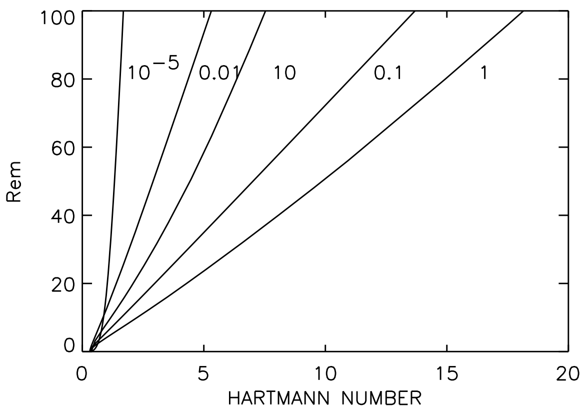

Fig. 9 The same as in Fig. 3 but for $\hat{\eta} = 0.05$. The curves are marked with the magnetic Prandtl numbers. Again rotating fluids with $\mathrm{Pm} = 1$ undergo the strongest stabilization.

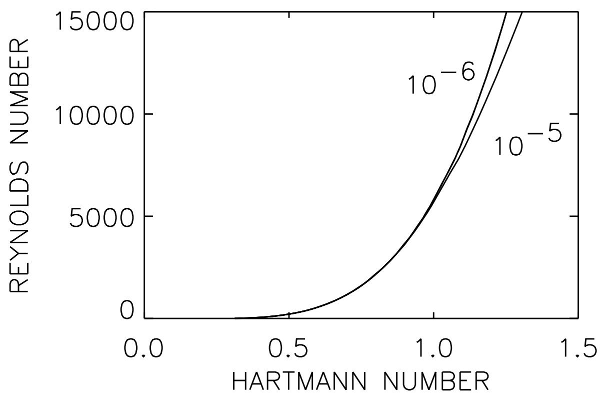

Fig. 10 The suppression of the TI in a wide-gap container by rigid rotation for $\mathrm{Pm} = 10^{-5}$ and $\mathrm{Pm} = 10^{-6}$. The standard Reynolds number (11) is given for experimental applications. There are no visible differences for the curves for $\mathrm{Pm} = 10^{-6}$ and for $\mathrm{Pm} = 0$. $\hat{\eta} = 0.05$, $\mu_B = 20$.

where R, B, and I are measured in centimeter, Gauss, and Ampere. It follows

$$I_{\mathrm{axis}} = \frac{5R_{\mathrm{in}}\mathrm{Ha}}{D}\sqrt{\mu_0\rho\nu\eta} = 5\sqrt{\frac{\hat{\eta}}{1-\hat{\eta}}}\mathrm{Ha}\sqrt{\mu_0\rho\nu\eta}. \quad (32)$$

With the numerical values for $\hat{\eta} = 0.05$ and $\sqrt{\mu_0\rho\nu\eta} = 25.6$ for the gallium-tin alloy used in the experiment PROMISE we find

$$I_{\mathrm{fluid}} = 11.8\,\mathrm{Ha} \quad [\mathrm{kA}] \quad (33)$$

for the electric current through the gallium and

$$I_{\mathrm{axis}} = 29.5\,\mathrm{Ha} \quad [\mathrm{A}] \quad (34)$$

for the current along the axis. With $\mathrm{Ha} = 0.31$ for $\hat{\eta} = 0.05$ (see Fig. 2) the results are $I_{\mathrm{fluid}} = 3.66$ kA and $I_{\mathrm{axis}} = 9.1$ A.

With the limit (23) of Ha for small $\hat{\eta}$ one finds

$$I_{\mathrm{fluid}} = 140\sqrt{1-\hat{\eta}}(1+\hat{\eta})\sqrt{\mu_0\rho\nu\eta} \quad [\mathrm{A}]. \quad (35)$$

For $\hat{\eta} \to 0$ the total current through the gallium becomes 3.50 kA which is within the present-day technical possibilities. Hence, we have shown that it should be possible to realize the nonaxisymmetric current-driven TI in the MHD laboratory also with fluids of small magnetic Prandtl number, e.g. with sodium or gallium.

5.2 Uniform rotation

Figure 9 gives for the wide-gap container the rotational quenching of the TI for various values of the magnetic Prandtl number similar to Fig. 3 for the narrow gap. Again the lines for marginal instability in the Rem-Ha plane are straight lines. The line for $\mathrm{Pm} = 1$ gives the maximum stabilization by rigid rotation. A stronger stabilization does not exist. Hence, if $\mathrm{Rem} \lesssim 4 \cdot \mathrm{Ha}$ then all fluids are unstable if subject to rotation. The rotational stabilization is much weaker for $\mathrm{Pm} \neq 1$. It is particularly weak for very small Pm.

The question is whether also the rotational stabilization of the TI can be probed in the laboratory. The rigidly rotating wide-gap container is thus considered also for the very small magnetic Prandtl numbers of fluid metals. The magnetic Prandtl number of sodium is about $\mathrm{Pm} = 10^{-5}$, and for gallium it is about $\mathrm{Pm} = 10^{-6}$. Without rotation the critical Hartmann number is 0.31 for this container (independent of Pm). With rotation the numerical results are given in Fig. 10. We find the rotational stabilization also existing for conducting fluids with their small Pm. For not too fast rotation the differences of the resulting critical Ha are very small for both the fluid conductors so that even experiments with gallium are also possible. For the rather small Reynolds number of order 1000 the marginal-stable magnetic field is about two times higher than for $\mathrm{Re} = 0$. It should thus not be too complicated to find the basic effect of the rotational suppression of TI – which proves to be important both for the rapid-rotating hot MS stars and also for neutron stars – in the MHD laboratory.

The curves in Fig. 10 do not become more steep for $\mathrm{Pm} \to 0$. There is no visible difference between the curves for $\mathrm{Pm} = 10^{-6}$ and $\mathrm{Pm} = 0$.

6 Conclusions

In this paper the interplay of toroidal magnetic fields and rotation for incompressible fluids of uniform density filling the gap between the cylinders of a Taylor-Couette container is considered. The toroidal field is the result of an electric current of homogeneous density. It is shown that for zero rotation the critical magnetic field amplitudes for marginal Tayler instability does not depend on the magnetic Prandtl number but the critical magnetic field strongly depends on the gap width. It is very high for small gaps and it this rather low for wide gaps. For small enough inner radius R_{in} the critical Hartmann number (of the inner field strength) runs as $R_{\rm in}^{1.5}$. The resulting electric currents necessary for TI with $m = 1$ slightly exceeds 3.5 kA if the material is the same gallium-tin alloy as used in the experiment PROMISE. Such currents can easily be produced in the laboratory.

For a narrow gap with $\hat{\eta} = 0.95$ the rotational quenching of the TI is studied in detail. Figure 3 displays the rotational stabilization for various magnetic Prandtl numbers. To normalize the basic rotation a 'mixed' Reynolds number (13) is used in which – as in the Hartmann number – the viscosities ν and η are symmetric. The ratio of this Reynolds number $\rm Rem$ and the Hartmann number (i.e. the magnetic Mach number) is free of any diffusivity. For fast enough rotation just this ratio describes the rotational quenching of TI for various $\rm Pm$. In this representation the results for $\rm Pm$ between 0.01 and 10 are rather simple. The most effective stabilization of TI happens for $\rm Pm = 1$. It is weaker for both smaller $\rm Pm$ and higher $\rm Pm$. This is an unexpected result which may warn that many numerical simulations with $\rm Pm \simeq 1$ could overlook the nonaxisymmetric instability of strong toroidal fields.

Also the inclusion of differential rotation leads to surprising results. For $\rm Pm = 1$ the Fig. 5 presents the basic differences for rotation laws with different signs of $\mathrm{d}\Omega/\mathrm{d}R$. While superrotation always stabilizes the magnetic field, there is a dramatic destabilization phenomenon by subrotation for medium rotation rates. This is the effect announced with Eq. (1) by Acheson (1978). Slow and fast rotators with negative shear thus have a very different stability behavior. However, if the rotation is too strong then the nonaxisymmetric instabilities are more and more destroyed by the very high shear amplitudes (see Fig. 6).

Our calculations demonstrate that rigid rotation always stabilizes the magnetic field against the nonaxisymmetric TI. We have also shown that this rotational stabilization can be realized in the laboratory. Figure 10 provides the result that the critical Hartmann number in a wide-gap container of $\hat{\eta} = 0.05$ can be increased by a factor of two by a rigid rotation of Reynolds number 1000. For the gallium-tin alloy with its molecular viscosity of 3.4×10^{-3} cm^2/s this Reynolds number is reached for a rotation frequency of about $11.4/R_{\rm out}^2$ Hz with $R_{\rm out}$ in cm. The rotation frequency of 0.11 Hz for $R_{\rm out} \simeq 10$ cm is rather small. For the same container one needs the electric current of 7.32 kAmp through the gallium-tin alloy to realize the Tayler instability in the rotating fluid conductor.

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