Taylor-Couette flow stability: effect of vertical density stratification and azimuthal magnetic fields

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Abstract. Also vertical density stratification and/or azimuthal magnetic fields essentially change the stability properties of the Taylor-Couette flow. Our results confirm the conclusion of Molemaker, McWilliams & Yavneh (2001) that vertically stratified Taylor-Couette flows are unstable against nonaxisymmetric disturbances if the angular velocity decreases with radius but not the angular momentum. Moreover, azimuthal magnetic fields can destabilize MHD Taylor-Couette flows with arbitrary rotation law for appropriate magnetic field geometry and amplitude. Such experiments in the laboratory might be much-promising cases due to the independence of the stability properties on the (very small) magnetic Prandtl number.

DENSITY-STRATIFIED TAYLOR-COUETTE FLOW

The Taylor-Couette flow stability is discussed here under the influence of vertical density stratifications or of azimuthal magnetic fields. A viscous Taylor-Couette flow between concentric rotating cylinders leads to the rotation law

$$\Omega(R) = a + \frac{b}{R^2}, \tag{1}$$

where a and b are the constants related to the angular velocities $\Omega_{\rm in}$ and $\Omega_{\rm out}$ with which the inner and the outer cylinders rotate. One finds

$$a = \Omega_{\rm in}\frac{\hat{\mu}-\hat{\eta}^2}{1-\hat{\eta}^2}, \qquad b = \Omega_{\rm in}R_{\rm in}^2\frac{1-\hat{\mu}}{1-\hat{\eta}^2} \tag{2}$$

with $R_{\rm in}$ and $R_{\rm out}$ as the radii of the two cylinders with

$$\hat{\mu} = \frac{\Omega_{\rm out}}{\Omega_{\rm in}} \tag{3}$$

and $\hat{\eta} = R_{\rm in}/R_{\rm out}$. According to the Rayleigh criterion the ideal flow is stable whenever the specific angular momentum increases outwards, i.e.

$$\hat{\mu} > \hat{\eta}^2. \tag{4}$$

If it is not too strong the magnetic field can play a destabilizing role and leads to the magnetorotational instability (MRI). In this regime the Rayleigh criterion for stability, (4), changes to

$$\hat{\mu} > 1. \tag{5}$$

Taylor-Couette flows with a stable axial density stratification were firstly studied by Thorpe (1966) who concluded that a stable stratification stabilizes the flow. With both experimental and theoretical studies Boubnov, Gledzer & Hopfinger (1995) and Caton, Janiaud & Hopfinger (2000) confirmed the stabilizing role of the stable stratification and showed that i) the critical Reynolds number depends on the buoyancy or Brunt-Väisälä frequency of the fluid and ii) the stratification reduces the height of the Taylor vortices. The numerical simulations of Hua, Le Gentil & Orlandi (1997) have reproduced these experimental results.

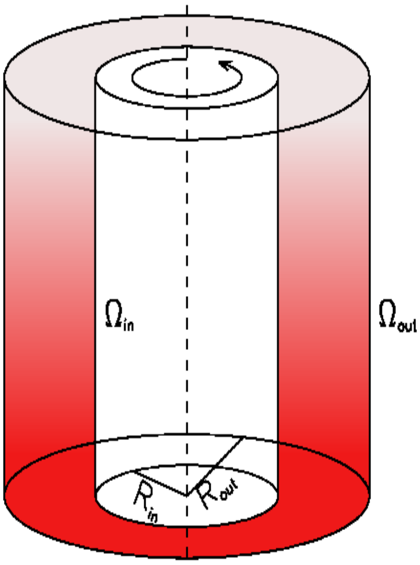

FIGURE 1. Taylor-Couette flow geometry with axial density stratification

The common feature of these studies is that the outer cylinder is at rest and the flow is unstable due to the Rayleigh condition (4) which was extended to stratified fluids by Ooyama (1966). Recently, Molemaker, McWilliams & Yavneh (2001) and Yavneh, McWilliams & Molemaker (2001) find that the sufficient condition for stability against nonaxisymmetric disturbances is the same as the condition (5) for magnetorotational stability of MHD Taylor-Couette flow. This result is due to a linear stability analysis for inviscid flow. The numerical results of Yavneh, McWilliams & Molemaker (2001) demonstrate that certain instabilities persist even for finite viscosity.

The results of Yavneh, McWilliams & Molemaker (2001) for viscous flows are rather illustrative. Here a more comprehensive study of viscous stratified Taylor-Couette flow is performed. The governing equations, the basic state and some restrictions of the usually used Boussinesq approximation are discussed in greater details.

In cylindrical coordinates $(R,\,\phi,\,z)$ the equations of incompressible stratified fluids with uniform dynamic viscosity, $\rho\nu=$ const, are

$$\begin{aligned}
&\frac{\partial u_R}{\partial t}+(\boldsymbol{u}\nabla)u_R-\frac{u_\phi^2}{R}=-\frac{1}{\rho}\frac{\partial P}{\partial R}+\nu\left[\Delta u_R-\frac{2}{R^2}\frac{\partial u_\phi}{\partial \phi}-\frac{u_R}{R^2}\right],\\
&\frac{\partial u_\phi}{\partial t}+(\boldsymbol{u}\nabla)u_\phi+\frac{u_\phi u_R}{R}=-\frac{1}{\rho R}\frac{\partial P}{\partial \phi}+\nu\left[\Delta u_\phi+\frac{2}{R^2}\frac{\partial u_R}{\partial \phi}-\frac{u_\phi}{R^2}\right],\\
&\frac{\partial u_z}{\partial t}+(\boldsymbol{u}\nabla)u_z=-\frac{1}{\rho}\frac{\partial P}{\partial z}-g+\nu\Delta u_z,\\
&\frac{\partial u_R}{\partial R}+\frac{u_R}{R}+\frac{1}{R}\frac{\partial u_\phi}{\partial \phi}+\frac{\partial u_z}{\partial z}=0,\\
&\frac{\partial \rho}{\partial t}+(\boldsymbol{u}\nabla)\rho=0,
\end{aligned}\tag{6}$$

where

$$(\boldsymbol{u}\nabla)u_R=u_R\frac{\partial u_R}{\partial R}+\frac{u_\phi}{R}\frac{\partial u_R}{\partial \phi}+u_z\frac{\partial u_R}{\partial z},\qquad \Delta u_R=\frac{\partial^2 u_R}{\partial R^2}+\frac{1}{R}\frac{\partial u_R}{\partial R}+\frac{1}{R^2}\frac{\partial^2 u_R}{\partial \phi^2}+\frac{\partial^2 u_R}{\partial z^2},\tag{7}$$

with $(u_R,\,u_\phi,\,u_z)$ as the velocity components, ρ the density, P the pressure, g the gravitational acceleration and ν the kinematic viscosity. We neglect the density diffusion term in the mass conservation equation due to the large value of the Schmidt number (the ratio of diffusive time to viscous time) in experiments (see e.g. Caton, Janiaud & Hopfinger 2000).

We are looking for the basic state with the prescribed velocity profile (1) and given density vertical stratification $\rho = \rho(z)$. The system (6) then takes the form

$$\frac{u_\phi^2}{R} = \frac{1}{\rho}\frac{\partial P}{\partial R}, \qquad \frac{\partial^2 u_\phi}{\partial R^2} + \frac{1}{R}\frac{\partial u_\phi}{\partial R} - \frac{u_\phi}{R^2} = 0, \qquad \frac{1}{\rho}\frac{\partial P}{\partial z} = -g. \tag{8}$$

The second equation of (8) defines the angular velocity profile (1). The remaining two equations are only possible if a certain integration condition is fulfilled.

Differentiating the first equation of (8) by z and the third equation by R, subtracting each other and using the supposed profiles of density and angular velocity we take the relation

$$R\Omega^2 \frac{d\rho}{dz} = 0. \tag{9}$$

The density can thus be a function of the vertical coordinate only without rotation ($\Omega = 0$); and the angular velocity can be a function of radius only without the vertical density stratification ($d\rho/dz = 0$). Thus, too simple profiles of the angular velocity, $\Omega = \Omega(R)$, and of the density $\rho = \rho(z)$ are not selfconsistent. For fixed inner and outer angular velocities we expect that the fluid rotation depends only on radial coordinate. We should admit, therefore, more general profiles for the density, i.e. $\rho = \rho(R,z)$, even though the initial density stratification for the fluid is only vertical. In this case, the condition (9) takes the form

$$R\Omega^2 \frac{\partial\rho}{\partial z} + g\frac{\partial\rho}{\partial R} = 0. \tag{10}$$

Under the centrifugal force the purely vertical stratification at the beginning transforms to a mixed vertical and radial stratification. This behavior strongly complicates the problem.

In real experiments the initial (without rotation) vertical stratification is small ($|d\log\rho/d\log z \ll 1|$) so as the ratio of centrifugal acceleration to the gravity acceleration

$$\left|\frac{R^2\Omega}{g}\right| \ll 1. \tag{11}$$

Then after (10) the radial stratification is also small. Let us, therefore, consider the ‘small-stratification case’ with

$$\rho = \rho_0 + \rho_1(R,z), \qquad \rho_1 \ll \rho_0, \tag{12}$$

where ρ_0 is the uniform background density and condition (10) is fulfilled at zero order. The overall state of the flow is thus described by

$$u'_R, \qquad u'_\phi + R\Omega(R), \qquad u'_z, \qquad P_0(R) + P_1(R,z) + P', \qquad \rho_0 + \rho_1(R,z) + \rho', \tag{13}$$

where $|P_1/P_0| \ll 1$ and u'_R, u'_ϕ, u'_z, P' and ρ' are the fluctuating quantities. Linearizing the system (6) and considering only the terms of the largest order then the system takes the Boussinesq form

$$\frac{\partial u_R}{\partial t} + \Omega\frac{\partial u_R}{\partial \phi} - 2\Omega u_\phi = -\frac{\partial}{\partial R}\left(\frac{P}{\rho_0}\right) + \nu_0\left[\Delta u_R - \frac{2}{R^2}\frac{\partial u_\phi}{\partial \phi} - \frac{u_R}{R^2}\right], \tag{14}$$

$$\frac{\partial u_\phi}{\partial t} + \Omega\frac{\partial u_\phi}{\partial \phi} + \frac{1}{R}\frac{\partial R^2\Omega}{\partial R}u_R = -\frac{1}{R}\frac{\partial}{\partial \phi}\left(\frac{P}{\rho_0}\right) + \nu_0\left[\Delta u_\phi + \frac{2}{R^2}\frac{\partial u_R}{\partial \phi} - \frac{u_\phi}{R^2}\right], \tag{15}$$

$$\frac{\partial u_z}{\partial t} + \Omega\frac{\partial u_z}{\partial \phi} = -\frac{\partial}{\partial z}\left(\frac{P}{\rho_0}\right) - g\frac{\rho}{\rho_0} + \nu_0\Delta u_z, \tag{16}$$

$$\frac{\partial}{\partial t}\left(\frac{\rho}{\rho_0}\right) + \Omega\frac{\partial}{\partial \phi}\left(\frac{\rho}{\rho_0}\right) - \frac{N^2}{g}u_z = 0, \tag{17}$$

$$\frac{\partial u_R}{\partial R} + \frac{u_R}{R} + \frac{1}{R}\frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_z}{\partial z} = 0, \tag{18}$$

where all the primes have been dropped again. N is the vertical Brunt-Väisälä frequency after

$$N^2 = -\frac{g}{\rho_0}\frac{\partial \rho_1}{\partial z}. \tag{19}$$

Due to (11) we neglect $\partial P_1/\partial R$ in (14) and $\partial \rho_1/\partial R$ in (15) arising from the radial stratification (they will be $|R^2\Omega/g|$ times smaller than terms arising from the vertical stratifications).

Here we suppose $\partial \rho_1/\partial z = \text{const}$ and then N^2 is a constant, too. In this case the coefficients of the system (18) only depend on the radial coordinate so that a normal mode expansion

$$F = F(R)\exp(\mathrm{i}(m\phi + kz + \omega t)) \tag{20}$$

can be used, where F represents any of the disturbed quantities.

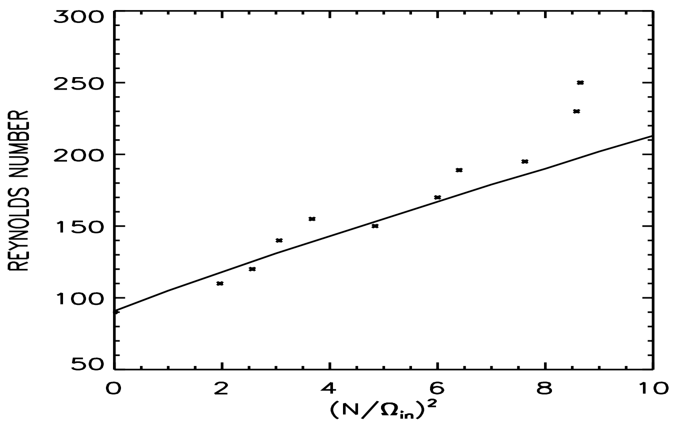

FIGURE 2. The marginal ($\omega = 0$) stability line for axisymmetric ($m = 0$) disturbances, $\hat{\eta} = 0.78$, $\hat{\mu} = 0$. The dots represent the experimental data of Boubnov, Gledzer & Hopfinger (1995).

Let $D = R_{\text{out}} - R_{\text{in}}$ be the gap between the cylinders. $R_0 = (R_{\text{in}}D)^{1/2}$ is used as the unit of length, the velocity $\Omega_{\text{in}}R_0$ as the unit of the perturbed velocity and Ω_{in} as the unit of ω, N and Ω. Using the same symbols for normalized quantities and redefining ρ as the nondimensional density $\rho g/\rho_0 R_0 \Omega_{\text{in}}^2$ and P as the nondimensional pressure $P/\rho_0\Omega_{\text{in}}^2 R_0^2$ we finally have

$$\frac{\partial^2 u_R}{\partial R^2} + \frac{1}{R}\frac{\partial u_R}{\partial R} - \frac{u_R}{R^2} - \left(k^2 + \frac{m^2}{R^2}\right)u_R - 2\mathrm{i}\frac{m}{R}u_\phi - \mathrm{i}\mathrm{Re}(\omega + m\Omega)u_R + 2\mathrm{Re}\Omega u_\phi - \mathrm{Re}\frac{\partial P}{\partial R} = 0, \tag{21}$$

$$\frac{\partial^2 u_\phi}{\partial R^2} + \frac{1}{R}\frac{\partial u_\phi}{\partial R} - \frac{u_\phi}{R^2} - \left(k^2 + \frac{m^2}{R^2}\right)u_\phi + 2\mathrm{i}\frac{m}{R}u_R - \mathrm{i}\mathrm{Re}(\omega + m\Omega)u_\phi - \mathrm{i}\mathrm{Re}\frac{m}{R}P - \frac{\mathrm{Re}}{R}\frac{\partial}{\partial R}(R^2\Omega) = 0, \tag{22}$$

$$\frac{\partial^2 u_z}{\partial R^2} + \frac{1}{R}\frac{\partial u_z}{\partial R} - \left(k^2 + \frac{m^2}{R^2}\right)u_z - \mathrm{i}\mathrm{Re}(\omega + m\Omega)u_z - \mathrm{i}\mathrm{Re}kP - \mathrm{Re}\rho = 0, \tag{23}$$

$$\mathrm{i}(\omega + m\Omega)\rho - N^2 u_z = 0, \tag{24}$$

$$\frac{\partial u_R}{\partial R} + \frac{u_R}{R} + \mathrm{i}\frac{m}{R}u_\phi + \mathrm{i}ku_z = 0. \tag{25}$$

The Reynolds number

$$\mathrm{Re} = \frac{\Omega_{\text{in}}R_0^2}{\nu} \tag{26}$$

and the Brunt-Väisälä frequency N are the only free parameters. The standard no-slip boundary conditions at the inner and outer cylinder are

$$u_R = u_\phi = u_z = 0. \tag{27}$$

The critical values of the Reynolds numbers above which the flow becomes unstable depend on the vertical wave number. They have a minimum at some wave number for fixed other parameters. This minimum value is called the critical Reynolds number. The same numerical method as in our previous papers on the MHD Taylor-Couette problem (see e.g. Rüdiger & Shalybkov 2002) was used. We only allow small negative imaginary parts of ω in the calculations to avoid problems with the corotation point $\omega = m\Omega$ for $m > 0$. The calculated critical Reynolds numbers are thus not for marginally stable state but for slightly unstable state. We have checked the existence of the transition from stable to unstable state for some arbitrary points.

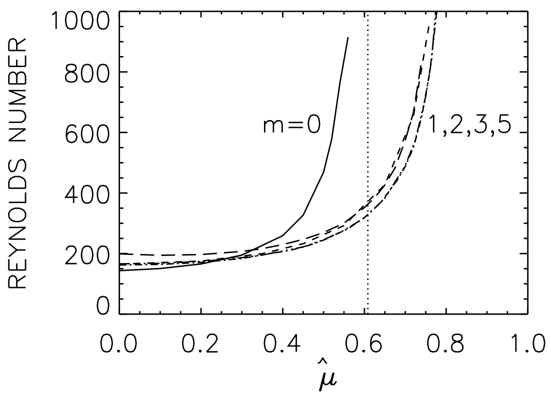

FIGURE 3. The marginal stability line for $m = 0$ and slightly overstable lines ($\Im(\omega) = -10^{-3}$) for $m > 0$ for $\hat{\eta} = 0.78$ and $N^2 = 4\Omega_{\rm in}^2$. The vertical line marks the value of $\hat{\eta}^2$.

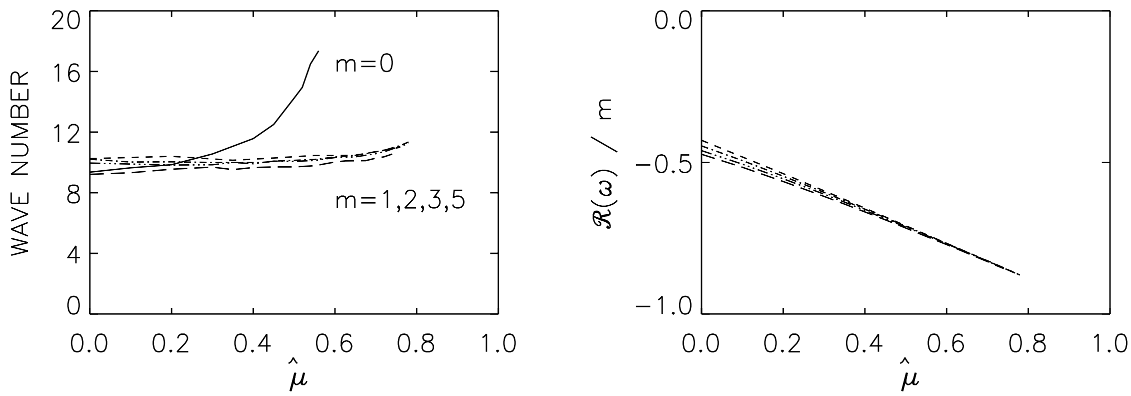

FIGURE 4. The same as in Fig. 3 but for vertical wave number (left) and $\Re(\omega)/m$ for $m = 1,2,3,4$ (right).

In Fig. 2 the calculated marginal stability line ($\omega = 0$) for axisymmetric disturbances is compared with experimental data (Boubnov, Gledzer & Hopfinger 1995). The agreement is very good except for large values of $N^2/\Omega_{\rm in}^2$. This disagreement may indicate the violation of the Boussinesq approximation.

The dependence of the critical Reynolds numbers on the ratio of inner and outer angular velocities, $\hat{\mu}$, is given in Fig. 3. The marginal stability line is plotted for $m = 0$ and slightly unstable lines ($\Im(\omega) = -10^{-3}$) for $m > 0$. The axisymmetric disturbances are unstable only for $\hat{\mu} < \hat{\eta}^2$ in accordance to the Rayleigh condition (4). Nevertheless, the nonaxisymmetric disturbances are also unstable into interval $\hat{\eta}^2 < \hat{\mu} < 1$.

For $m > 0$ the critical Reynolds numbers only slightly depend on m such as also the vertical wave numbers and the real part of ω divided by m ($\Re(\omega)/m$, see Fig. 4). The vertical wave number does not depend on $\hat{\mu}$ and $\Re(\omega)/m$ is nearly linear in $\hat{\mu}$.

With our normalization the vertical extension of the Taylor vortices is given by

$$\frac{\delta z}{R_{\mathrm{out}} - R_{\mathrm{in}}} = \frac{\pi}{k}\sqrt{\frac{\hat{\eta}}{1-\hat{\eta}}}. \tag{28}$$

It is order of unity for $N^2 = 4\Omega_{\mathrm{in}}^2$ and it decreases with increasing N^2.

Velocity eigenfunctions are presented in Fig. 5 for $m = 1$ and $\hat{\mu} = 0.7$ which exceeds $\hat{\eta}^2$. These functions are smooth and they do not exhibit any striking features near the boundaries.

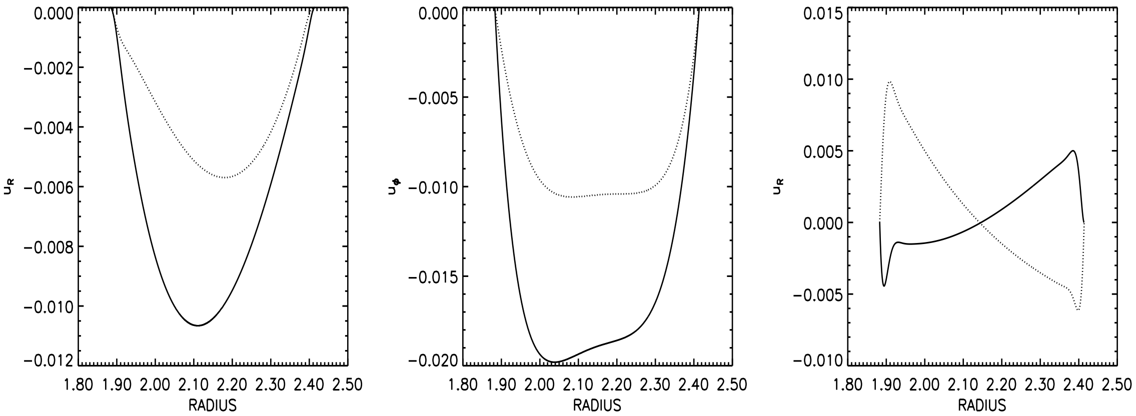

FIGURE 5. The velocity eigenfunctions for $m = 1$, $\hat{\eta} = 0.78$, $\hat{\mu} = 0.7$, $N^2 = 4\Omega_{\mathrm{in}}^2$ at the critical Reynolds number. The dotted lines are the real part and the solid lines are the imaginary parts of the frequency ω.

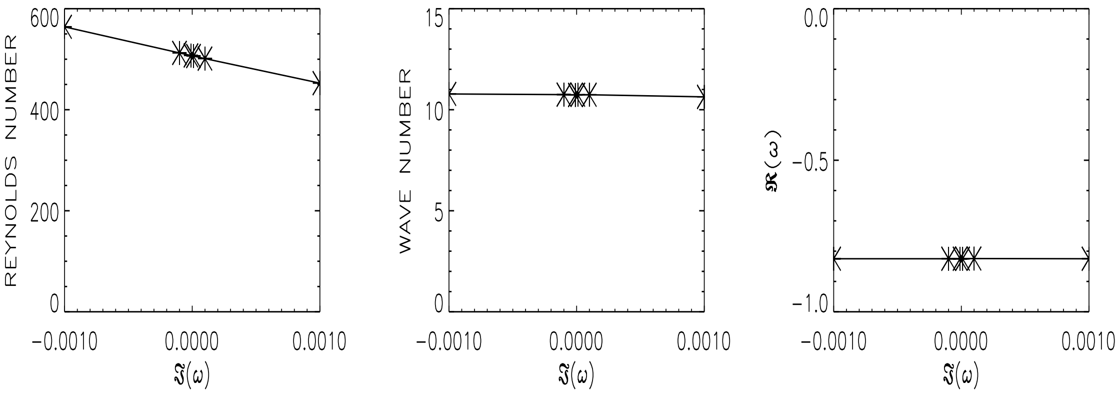

FIGURE 6. The Reynolds number Re, the vertical wave number k and the real part $\Re(\omega)$ of the eigenfrequency ω (from left to right) for the transition from positive to negative imaginary part of ω for $m = 1$, $\hat{\eta} = 0.78$, $\hat{\mu} = 0.7$, $N^2 = 4\Omega_{\mathrm{in}}^2$.

Figure 6 presents the behavior of the Reynolds numbers, the vertical wave numbers and the real part of ω for the transition region from positive to negative values of $\Im(\omega)$. We find a *continuous transition* around the marginal stability line. It is thus possible to realize the transition from the stable to the unstable flow regime in the laboratory. The vertical wave numbers and $\Re(\omega)$ do not vary but the Reynolds numbers show a surprisingly clear tendency around its critical value.

AZIMUTHAL FIELD

The Taylor-Couette flow stability in the presence of azimuthal magnetic fields is by far not so clear as it is for vertical magnetic fields. Michael (1954) found the necessary and sufficient condition for stability of ideal flows against axisymmetric disturbances. Edmonds (1958) considered dissipative fluids but with restrictive approximations: only conducting boundaries, absence of electrical currents and only for the narrow-gap limit. The absence of the electrical currents together with the conducting boundary conditions *always* leads to a suppression of the instability by the toroidal fields. The importance of the problem in astrophysics stimulated several discussions for ideal flows (Curry & Pudritz 1995) or in the frame of the local approximation (Menou, Balbus & Spruit 2004, Pessah & Psaltis 2004).

Here we discuss the linear stability of nonideal Taylor-Couette flows with azimuthal magnetic field. For simplicity only axisymmetric disturbances are considered. The equations of the problem can be found elsewhere (Chandrasekhar 1961) and they are not reproduced here. The basic solution is

$$U_R = U_z = B_R = B_z = 0, \qquad B_\phi = a_B R + \frac{b_B}{R}, \qquad U_\phi = R\Omega = aR + \frac{b}{R}, \tag{29}$$

where the constants a, b, a_B, b_B are defined by boundary conditions. The angular velocity constants are given by (2). The magnetic field constants are defined by the values of the azimuthal magnetic field $B_{\rm in}$ at the inner and the outer cylinders, i.e.

$$a_B = \frac{B_{\rm in}}{R_{\rm in}} \frac{\hat{\eta}(\hat{\mu}_B - \hat{\eta})}{1-\hat{\eta}^2}, \qquad b_B = B_{\rm in} R_{\rm in} \frac{1-\hat{\mu}_B\hat{\eta}}{1-\hat{\eta}^2}, \qquad \hat{\mu}_B = \frac{B_{\rm out}}{B_{\rm in}}. \tag{30}$$

The constants $B_{\rm in}$ and $B_{\rm out}$ are defined by the vertical electric currents inside the cylinders.

We are interested in the stability of the basic solution (29). As a first step the linear stability problem is only considered for axisymmetric perturbations. The linearized equations can be found in the book of Chandrasekhar (1961). The dimensionless numbers of the problem are the same as before with $B_{\rm in}$ instead B_0 within the Hartmann number definition, i.e.

$$\text{Ha} = \frac{B_{\rm in}\sqrt{(R_{\rm out}-R_{\rm in})R_{\rm in}}}{\sqrt{\mu_0 \rho \nu \eta}}. \tag{31}$$

The boundary conditions are given by (27) and the known magnetic conditions for isolating and for conducting cylinders.

According to Michael (1954) the necessary and sufficient condition for the stability of ideal flows is

$$\frac{1}{R^3}\frac{d}{dR}(R^2\Omega)^2 - \frac{R}{\mu_0\rho}\frac{d}{dR}\left(\frac{B_\phi}{R}\right)^2 > 0. \tag{32}$$

The condition (32) is known (in the absence of rotation) in the theory of pinch stability (see e.g. Kadomcev 1963). With (29) one finds

$$4a^2 + 4\frac{ab}{R^2} + \frac{V_{\rm A}^2}{U_{\rm in}^2}\left(4\frac{a_B b_B}{R^2} + 4\frac{b_B^2}{R^4}\right) > 0, \tag{33}$$

where $V_{\rm A}^2 = B_{\rm in}^2/\mu_0\rho$ is the Alfvén velocity and $U_{\rm in} = R_{\rm in}\Omega_{\rm in}$. The nonmagnetic part of (33) is positive when (4) is fulfilled. The magnetic field stabilizes the flow (the sum of the last two terms are positive) if

$$0 < \hat{\mu}_B < \frac{1}{\hat{\eta}}. \tag{34}$$

It is obvious that for negative magnetic part in (33) (i.e. $0 > \hat{\mu}_B > 1/\hat{\eta}$) one can always find such values of the magnetic field amplitude that (33) will be negative for any value of $\hat{\mu}$. Thus the magnetic field with the profile (29) can destabilize any ideal flow. Large $V_{\rm A}/U_{\rm in}$ do favor instability (if (34) is fulfilled) and this means large magnetic field and/or slow rotation.

Here the results for containers with resting outer cylinder and with a radius ratio of $\hat{\eta} = 0.5$ are presented. Note that the results do *not* depend on the magnetic Prandtl number. In Fig. 7 the critical Reynolds numbers are given as function of the Hartmann number for both insulating and conducting cylinders.

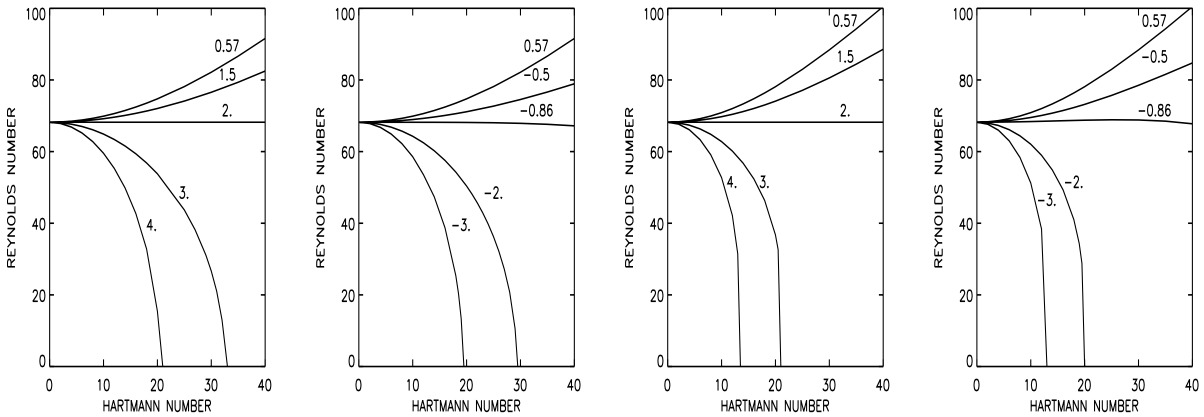

FIGURE 7. The lines of marginal stability for magnetic Taylor-Couette flow with *insulating* cylinders (left two plots) and *conducting* cylinders (right two plots) for $\hat{\eta} = 0.5$ and $\hat{\mu} = 0$. The lines are labeled by the $\hat{\mu}_B$ values.

The critical Reynolds numbers even vanish for some (critical) value of Ha for appropriate values of $\hat{\mu}_B$. Thus the flow becomes unstable without any rotation. This behavior is a direct manifestation of the pinch instability mentioned above. After Fig. 7 it seems that the interval of $\hat{\mu}_B$ where magnetic fields only suppress the instability is not the same as (34) for ideal flows. The calculations with larger Ha number show that these intervals are really the same (see also Fig. 8).

The critical Reynolds numbers are systematically higher for conducting cylinders for fixed Ha and fixed $\hat{\mu}_B$ from (34) and systematically lower for other $\hat{\mu}_B$. The minimum Hartmann number for $\mathrm{Re} = 0$ is thus larger for insulating cylinders for fixed $\hat{\mu}_B$.

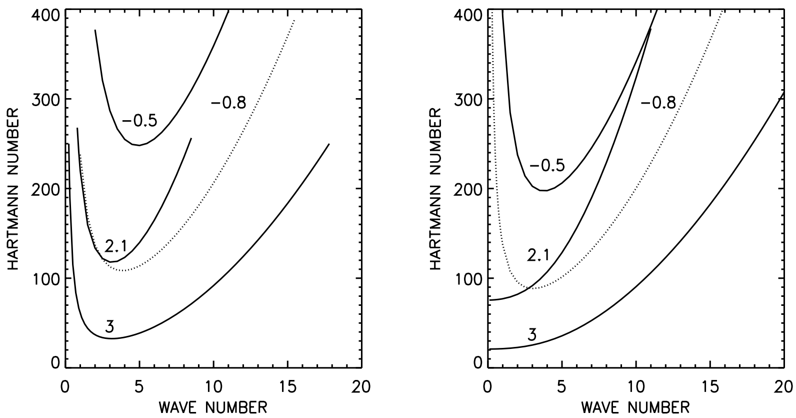

FIGURE 8. The lines where $\mathrm{Re} = 0$ for insulating (left) and conducting (right) cylinders for $\hat{\eta} = 0.5$. The lines are labeled by $\hat{\mu}_B$ values.

The vertical wave numbers increase with increasing Hartmann numbers for stabilizing magnetic field. This increasing is sharp for conducting cylinders and slow for insulating ones. The behavior of the vertical wave number is much more complicated for magnetic fields destabilizing the flow and also for Hartmann numbers larger than their critical values.

The relations for the wave numbers and the Hartmann numbers for $\mathrm{Re} = 0$ are given by Fig. 8. They do obviously not depend on the rotation parameter $\hat{\mu}$. For hydrodynamically stable flow there are solutions only above the lines. This means that solutions exist only for wave number inside some interval with $\hat{\mu}_B < 0$ and for wave number smaller than some critical value for conducting cylinders and $\hat{\mu}_B > 1/\hat{\eta}$. For hydrodynamically unstable flow there are three or two branches of the solution separated by the critical wave numbers. The interval where solutions exist for hydrodynamically stable flow increases with increasing Hartmann number. Such a behavior also Pessah & Psaltis (2004) found with a local stability analysis for an ideal fluid.

Figure 9 shows the eigenfunctions for minimum critical Hartmann number for insulating cylinders and for $\hat{\mu}_B = 3$. The Hartmann number and the wave number are $\mathrm{Ha} = 32.6$ and $k = 3.13$. The disturbed state has only a toroidal magnetic field component and has both radial and vertical velocities constituting the classical Taylor vortices.

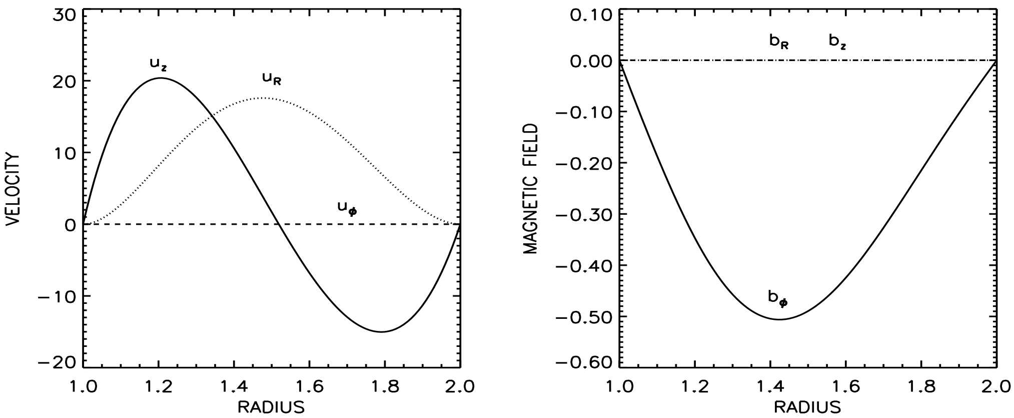

FIGURE 9. The eigenfunctions for velocity (left) and magnetic field (right) for insulating cylinders and $\hat{\eta} = 0.5$, $\hat{\mu}_B = 3$, $\mathrm{Ha} = 32.6$, $\mathrm{Re} = 0$, $k = 3.13$. The Ha value is minimum just for this $\hat{\mu}_B$ (see Fig. 8).

Taking the parameter values for liquid sodium ($\nu = 7.1 \cdot 10^{-3}$ cm^2/s, $\eta = 810$ cm^2/s, $\rho = 0.92$ g/cm^3) and typical dimensions such as $R_{\mathrm{in}} = 10$ cm, $R_{\mathrm{out}} = 20$ cm and $\mathrm{Ha}^2 = 10^3$ (see Fig. 7) we get for $\mathrm{Re} = 0$ the magnetic field on the inner cylinder as only about 30 Gauss.

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