

## CITED REFERENCES AND FURTHER READING:

- Bell, T.C., Cleary, J.G., and Witten, I.H. 1990, *Text Compression* (Englewood Cliffs, NJ: Prentice-Hall).
- Nelson, M. 1991, *The Data Compression Book* (Redwood City, CA: M&T Books).
- Witten, I.H., Neal, R.M., and Cleary, J.G. 1987, *Communications of the ACM*, vol. 30, pp. 520–540. [1]

## 20.6 Arithmetic at Arbitrary Precision

Let's compute the number  $\pi$  to a couple of thousand decimal places. In doing so, we'll learn some things about multiple precision arithmetic on computers and meet quite an unusual application of the fast Fourier transform (FFT). We'll also develop a set of routines that you can use for other calculations at any desired level of arithmetic precision.

To start with, we need an analytic algorithm for  $\pi$ . Useful algorithms are quadratically convergent, i.e., they double the number of significant digits at each iteration. Quadratically convergent algorithms for  $\pi$  are based on the *AGM* (*arithmetic geometric mean*) method, which also finds application to the calculation of elliptic integrals (cf. §6.1.1) and in advanced implementations of the ADI method for elliptic partial differential equations (§19.5). Borwein and Borwein [1] treat this subject, which is beyond our scope here. One of their algorithms for  $\pi$  starts with the initializations

$$\begin{aligned} X_0 &= \sqrt{2} \\ \pi_0 &= 2 + \sqrt{2} \\ Y_0 &= \sqrt[4]{2} \end{aligned} \tag{20.6.1}$$

and then, for  $i = 0, 1, \dots$ , repeats the iteration

$$\begin{aligned} X_{i+1} &= \frac{1}{2} \left( \sqrt{X_i} + \frac{1}{\sqrt{X_i}} \right) \\ \pi_{i+1} &= \pi_i \left( \frac{X_{i+1} + 1}{Y_i + 1} \right) \\ Y_{i+1} &= \frac{Y_i \sqrt{X_{i+1}} + \frac{1}{\sqrt{X_{i+1}}}}{Y_i + 1} \end{aligned} \tag{20.6.2}$$

The value  $\pi$  emerges as the limit  $\pi_\infty$ .

Now, to the question of how to do arithmetic to arbitrary precision: In a high-level language like C, a natural choice is to work in radix (base) 256, so that character arrays can be directly interpreted as strings of digits. At the very end of our calculation, we will want to convert our answer to radix 10, but that is essentially a frill for the benefit of human ears, accustomed to the familiar chant, “three point

one four one five nine. . .” For any less frivolous calculation, we would likely never leave base 256 (or the thence trivially reachable hexadecimal, octal, or binary bases).

We will adopt the convention of storing digit strings in the “human” ordering, that is, with the first stored digit in an array being most significant, the last stored digit being least significant. The opposite convention would, of course, also be possible. “Carries,” where we need to partition a number larger than 255 into a low-order byte and a high-order carry, present a minor programming annoyance, solved, in the routines below, by the use of the macros LOBYTE and HIBYTE.

It is easy at this point, following Knuth [2], to write a routine for the “fast” arithmetic operations: short addition (adding a single byte to a string), addition, subtraction, short multiplication (multiplying a string by a single byte), short division, ones-complement negation; and a couple of utility operations, copying and left-shifting strings. (On the diskette, these functions are all in the single file mpops.c.)

```
#define LOBYTE(x) ((unsigned char) ((x) & 0xff))
#define HIBYTE(x) ((unsigned char) ((x) >> 8 & 0xff))
```

Multiple precision arithmetic operations done on character strings, interpreted as radix 256 numbers. This set of routines collects the simpler operations.

```
void mpadd(unsigned char w[], unsigned char u[], unsigned char v[], int n)
Adds the unsigned radix 256 integers u[1..n] and v[1..n] yielding the unsigned integer w[1..n+1].
```

```
{
    int j;
    unsigned short ireg=0;

    for (j=n;j>=1;j--) {
        ireg=u[j]+v[j]+HIBYTE(ireg);
        w[j+1]=LOBYTE(ireg);
    }
    w[1]=HIBYTE(ireg);
}
```

```
void mpsub(int *is, unsigned char w[], unsigned char u[], unsigned char v[],
int n)
```

Subtracts the unsigned radix 256 integer v[1..n] from u[1..n] yielding the unsigned integer w[1..n]. If the result is negative (wraps around), is is returned as -1; otherwise it is returned as 0.

```
{
    int j;
    unsigned short ireg=256;

    for (j=n;j>=1;j--) {
        ireg=255+u[j]-v[j]+HIBYTE(ireg);
        w[j]=LOBYTE(ireg);
    }
    *is=HIBYTE(ireg)-1;
}
```

```
void mpsad(unsigned char w[], unsigned char u[], int n, int iv)
```

Short addition: the integer iv (in the range  $0 \leq iv \leq 255$ ) is added to the unsigned radix 256 integer u[1..n], yielding w[1..n+1].

```
{
    int j;
    unsigned short ireg;

    ireg=256*iv;
```

```

    for (j=n;j>=1;j--) {
        ireg=u[j]+HIBYTE(ireg);
        w[j+1]=LOBYTE(ireg);
    }
    w[1]=HIBYTE(ireg);
}

```

void mpsmu(unsigned char w[], unsigned char u[], int n, int iv)  
Short multiplication: the unsigned radix 256 integer u[1..n] is multiplied by the integer iv (in the range  $0 \leq iv \leq 255$ ), yielding w[1..n+1].

```

{
    int j;
    unsigned short ireg=0;

    for (j=n;j>=1;j--) {
        ireg=u[j]*iv+HIBYTE(ireg);
        w[j+1]=LOBYTE(ireg);
    }
    w[1]=HIBYTE(ireg);
}

```

void mpsdv(unsigned char w[], unsigned char u[], int n, int iv, int \*ir)  
Short division: the unsigned radix 256 integer u[1..n] is divided by the integer iv (in the range  $0 \leq iv \leq 255$ ), yielding a quotient w[1..n] and a remainder ir (with  $0 \leq ir \leq 255$ ).

```

{
    int i,j;

    *ir=0;
    for (j=1;j<=n;j++) {
        i=256*(ir)+u[j];
        w[j]=(unsigned char) (i/iv);
        *ir=i % iv;
    }
}

```

void mpneg(unsigned char u[], int n)  
Ones-complement negate the unsigned radix 256 integer u[1..n].

```

{
    int j;
    unsigned short ireg=256;

    for (j=n;j>=1;j--) {
        ireg=255-u[j]+HIBYTE(ireg);
        u[j]=LOBYTE(ireg);
    }
}

```

void mpmov(unsigned char u[], unsigned char v[], int n)  
Move v[1..n] onto u[1..n].

```

{
    int j;

    for (j=1;j<=n;j++) u[j]=v[j];
}

```

void mplsh(unsigned char u[], int n)  
Left shift u(2..n+1) onto u[1..n].

```

{
    int j;

    for (j=1;j<=n;j++) u[j]=u[j+1];
}

```

Full multiplication of two digit strings, if done by the traditional hand method, is not a fast operation: In multiplying two strings of length  $N$ , the multiplicand would be short-multiplied in turn by each byte of the multiplier, requiring  $O(N^2)$  operations in all. We will see, however, that *all* the arithmetic operations on numbers of length  $N$  can in fact be done in  $O(N \times \log N \times \log \log N)$  operations.

The trick is to recognize that multiplication is essentially a *convolution* (§13.1) of the digits of the multiplicand and multiplier, followed by some kind of carry operation. Consider, for example, two ways of writing the calculation  $456 \times 789$ :

$$\begin{array}{r}
 456 \\
 \times 789 \\
 \hline
 4104 \\
 3648 \\
 3192 \\
 \hline
 359784
 \end{array}
 \qquad
 \begin{array}{r}
 4 \ 5 \ 6 \\
 \times 7 \ 8 \ 9 \\
 \hline
 36 \ 45 \ 54 \\
 32 \ 40 \ 48 \\
 28 \ 35 \ 42 \\
 \hline
 28 \ 67 \ 118 \ 93 \ 54 \\
 \hline
 3 \ 5 \ 9 \ 7 \ 8 \ 4
 \end{array}$$

The tableau on the left shows the conventional method of multiplication, in which three separate short multiplications of the full multiplicand (by 9, 8, and 7) are added to obtain the final result. The tableau on the right shows a different method (sometimes taught for mental arithmetic), where the single-digit cross products are all computed (e.g.  $8 \times 6 = 48$ ), then added in columns to obtain an incompletely carried result (here, the list 28, 67, 118, 93, 54). The final step is a single pass from right to left, recording the single least-significant digit and carrying the higher digit or digits into the total to the left (e.g.  $93 + 5 = 98$ , record the 8, carry 9).

You can see immediately that the column sums in the right-hand method are components of the convolution of the digit strings, for example  $118 = 4 \times 9 + 5 \times 8 + 6 \times 7$ . In §13.1 we learned how to compute the convolution of two vectors by the fast Fourier transform (FFT): Each vector is FFT'd, the two complex transforms are multiplied, and the result is inverse-FFT'd. Since the transforms are done with floating arithmetic, we need sufficient precision so that the exact integer value of each component of the result is discernible in the presence of roundoff error. We should therefore allow a (conservative) few times  $\log_2(\log_2 N)$  bits for roundoff in the FFT. A number of length  $N$  bytes in radix 256 can generate convolution components as large as the order of  $(256)^2 N$ , thus requiring  $16 + \log_2 N$  bits of precision for exact storage. If  $it$  is the number of bits in the floating mantissa (cf. §20.1), we obtain the condition

$$16 + \log_2 N + \text{few} \times \log_2 \log_2 N < it \quad (20.6.3)$$

We see that single precision, say with  $it = 24$ , is inadequate for any interesting value of  $N$ , while double precision, say with  $it = 53$ , allows  $N$  to be greater than  $10^6$ , corresponding to some millions of decimal digits. The following routine therefore presumes double precision versions of `realft` (§12.3) and `four1` (§12.2), here called `drealft` and `dfour1`. (These routines are included on the *Numerical Recipes* diskettes.)

```

#include "nrutil.h"
#define RX 256.0

void mpmul(unsigned char w[], unsigned char u[], unsigned char v[], int n,
           int m)
Uses Fast Fourier Transform to multiply the unsigned radix 256 integers u[1..n] and v[1..m],
yielding a product w[1..n+m].
{
    void drealft(double data[], unsigned long n, int isign);    double version of realft.
    int j,mn,nn=1;
    double cy,t,*a,*b;

    mn=IMAX(m,n);
    while (nn < mn) nn <<= 1;    Find the smallest usable power of two for the trans-
    nn <<= 1;                    form.
    a=dvector(1,nn);
    b=dvector(1,nn);
    for (j=1;j<=n;j++)    Move U to a double precision floating array.
        a[j]=(double)u[j];
    for (j=n+1;j<=nn;j++) a[j]=0.0;
    for (j=1;j<=m;j++)    Move V to a double precision floating array.
        b[j]=(double)v[j];
    for (j=m+1;j<=nn;j++) b[j]=0.0;
    drealft(a,nn,1);    Perform the convolution: First, the two Fourier trans-
    drealft(b,nn,1);    forms.
    b[1] *= a[1];    Then multiply the complex results (real and imagi-
    b[2] *= a[2];    nary parts).
    for (j=3;j<=nn;j+=2) {
        b[j]=(t=b[j])*a[j]-b[j+1]*a[j+1];
        b[j+1]=t*a[j+1]+b[j+1]*a[j];
    }
    drealft(b,nn,-1);    Then do the inverse Fourier transform.
    cy=0.0;    Make a final pass to do all the carries.
    for (j=nn;j>=1;j--) {
        t=b[j]/(nn>>1)+cy+0.5;    The 0.5 allows for roundoff error.
        cy=(unsigned long) (t/RX);
        b[j]=t-cy*RX;
    }
    if (cy >= RX) nrerror("cannot happen in fftmul");
    w[1]=(unsigned char) cy;    Copy answer to output.
    for (j=2;j<=n+m;j++)
        w[j]=(unsigned char) b[j-1];
    free_dvector(b,1,nn);
    free_dvector(a,1,nn);
}

```

With multiplication thus a “fast” operation, division is best performed by multiplying the dividend by the reciprocal of the divisor. The reciprocal of a value  $V$  is calculated by iteration of Newton’s rule,

$$U_{i+1} = U_i(2 - VU_i) \quad (20.6.4)$$

which results in the quadratic convergence of  $U_\infty$  to  $1/V$ , as you can easily prove. (Many supercomputers and RISC machines actually use this iteration to perform divisions.) We can now see where the operations count  $N \log N \log \log N$ , mentioned above, originates:  $N \log N$  is in the Fourier transform, with the iteration to converge Newton’s rule giving an additional factor of  $\log \log N$ .

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```

#include "nrutil.h"
#define MF 4
#define BI (1.0/256)

void mpinv(unsigned char u[], unsigned char v[], int n, int m)
Character string v[1..m] is interpreted as a radix 256 number with the radix point after
(nonzero) v[1]; u[1..n] is set to the most significant digits of its reciprocal, with the radix
point after u[1].
{
    void mpmov(unsigned char u[], unsigned char v[], int n);
    void mpmul(unsigned char w[], unsigned char u[], unsigned char v[], int n,
               int m);
    void mpneg(unsigned char u[], int n);
    unsigned char *rr,*s;
    int i,j,maxmn,mm;
    float fu,fv;

    maxmn=IMAX(n,m);
    rr=cvector(1,1+(maxmn<<1));
    s=cvector(1,maxmn);
    mm=IMIN(MF,m);
    fv=(float) v[mm];
    for (j=mm-1;j>=1;j--) {           Use ordinary floating arithmetic to get an initial ap-
        fv *= BI;                       proximation.
        fv += v[j];
    }
    fu=1.0/fv;
    for (j=1;j<=n;j++) {
        i=(int) fu;
        u[j]=(unsigned char) i;
        fu=256.0*(fu-i);
    }
    for (;;) {                         Iterate Newton's rule to convergence.
        mpmul(rr,u,v,n,m);               Construct 2 - UV in S.
        mpmov(s,&rr[1],n);
        mpneg(s,n);
        s[1] -= 254;                       Multiply SU into U.
        mpmul(rr,s,u,n,n);
        mpmov(u,&rr[1],n);
        for (j=2;j<n;j++)                 If fractional part of S is not zero, it has not converged
            if (s[j]) break;               to 1.
        if (j==n) {
            free_cvector(s,1,maxmn);
            free_cvector(rr,1,1+(maxmn<<1));
            return;
        }
    }
}

```

Division now follows as a simple corollary, with only the necessity of calculating the reciprocal to sufficient accuracy to get an exact quotient and remainder.

```

#include "nrutil.h"
#define MACC 6

void mpdiv(unsigned char q[], unsigned char r[], unsigned char u[],
           unsigned char v[], int n, int m)
Divides unsigned radix 256 integers u[1..n] by v[1..m] (with  $m \leq n$  required), yielding a
quotient q[1..n-m+1] and a remainder r[1..m].
{
    void mpinv(unsigned char u[], unsigned char v[], int n, int m);
    void mpmov(unsigned char u[], unsigned char v[], int n);

```

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```

void mpmul(unsigned char w[], unsigned char u[], unsigned char v[], int n,
           int m);
void mpsad(unsigned char w[], unsigned char u[], int n, int iv);
void mpsub(int *is, unsigned char w[], unsigned char u[], unsigned char v[],
           int n);
int is;
unsigned char *rr,*s;

rr=cvector(1,(n+MACC)<<1);
s=cvector(1,(n+MACC)<<1);
mpinv(s,v,n+MACC,m);           Set S = 1/V.
mpmul(rr,s,u,n+MACC,n);       Set Q = SU.
mpsad(s,rr,n+n+MACC/2,1);
mpmov(q,&s[2],n-m+1);
mpmul(rr,q,v,n-m+1,m);       Multiply and subtract to get the remainder.
mpsub(&is,&rr[1],u,&rr[1],n);
if (is) nrerror("MACC too small in mpdiv");
mpmov(r,&rr[n-m+1],m);
free_cvector(s,1,(n+MACC)<<1);
free_cvector(rr,1,(n+MACC)<<1);
}

```

Square roots are calculated by a Newton's rule much like division. If

$$U_{i+1} = \frac{1}{2}U_i(3 - VU_i^2) \quad (20.6.5)$$

then  $U_\infty$  converges quadratically to  $1/\sqrt{V}$ . A final multiplication by  $V$  gives  $\sqrt{V}$ .

```

#include <math.h>
#include "nrutil.h"
#define MF 3
#define BI (1.0/256)

void mpsqrt(unsigned char w[], unsigned char u[], unsigned char v[], int n,
            int m)
Character string v[1..m] is interpreted as a radix 256 number with the radix point after v[1];
w[1..n] is set to its square root (radix point after w[1]), and u[1..n] is set to the reciprocal
thereof (radix point before u[1]). w and u need not be distinct, in which case they are set
to the square root.
{
    void mplsh(unsigned char u[], int n);
    void mpmov(unsigned char u[], unsigned char v[], int n);
    void mpmul(unsigned char w[], unsigned char u[], unsigned char v[], int n,
               int m);
    void mpneg(unsigned char u[], int n);
    void mpsdv(unsigned char w[], unsigned char u[], int n, int iv, int *ir);
    int i,ir,j,mm;
    float fu,fv;
    unsigned char *r,*s;

    r=cvector(1,n<<1);
    s=cvector(1,n<<1);
    mm=IMIN(m,MF);
    fv=(float) v[mm];           Use ordinary floating arithmetic to get an initial ap-
    for (j=mm-1;j>=1;j--) {     proximation.
        fv *= BI;
        fv += v[j];
    }
    fu=1.0/sqrt(fv);
    for (j=1;j<=n;j++) {
        i=(int) fu;
        u[j]=(unsigned char) i;
    }
}

```

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```

    fu=256.0*(fu-i);
}
for (;;) {
    mpmul(r,u,u,n,n);          Iterate Newton's rule to convergence.
                              Construct  $S = (3 - VU^2)/2$ .
    mplsh(r,n);
    mpmul(s,r,v,n,IMIN(m,n));
    mplsh(s,n);
    mpneg(s,n);
    s[1] -= 253;
    mpsdv(s,s,n,2,&ir);
    for (j=2;j<n;j++) {
        if (s[j]) {           If fractional part of  $S$  is not converged
                              to 1.
            mpmul(r,s,u,n,n);   Replace  $U$  by  $SU$ .
            mpmov(u,&r[1],n);
            break;
        }
    }
    if (j<n) continue;
    mpmul(r,u,v,n,IMIN(m,n));   Get square root from reciprocal and return.
    mpmov(w,&r[1],n);
    free_cvector(s,1,n<<1);
    free_cvector(r,1,n<<1);
    return;
}
}

```

We already mentioned that radix conversion to decimal is a merely cosmetic operation that should normally be omitted. The simplest way to convert a fraction to decimal is to multiply it repeatedly by 10, picking off (and subtracting) the resulting integer part. This, has an operations count of  $O(N^2)$ , however, since each liberated decimal digit takes an  $O(N)$  operation. It is possible to do the radix conversion as a fast operation by a “divide and conquer” strategy, in which the fraction is (fast) multiplied by a large power of 10, enough to move about half the desired digits to the left of the radix point. The integer and fractional pieces are now processed independently, each further subdivided. If our goal were a few billion digits of  $\pi$ , instead of a few thousand, we would need to implement this scheme. For present purposes, the following lazy routine is adequate:

```

#define IAZ 48

void mp2dfr(unsigned char a[], unsigned char s[], int n, int *m)
Converts a radix 256 fraction a[1..n] (radix point before a[1]) to a decimal fraction represented as an ascii string s[1..m], where m is a returned value. The input array a[1..n] is destroyed. NOTE: For simplicity, this routine implements a slow ( $\propto N^2$ ) algorithm. Fast ( $\propto N \ln N$ ), more complicated, radix conversion algorithms do exist.
{
    void mplsh(unsigned char u[], int n);
    void mpsmu(unsigned char w[], unsigned char u[], int n, int iv);
    int j;

    *m=(int) (2.408*n);
    for (j=1;j<=*m;j++) {
        mpsmu(a,a,n,10);
        s[j]=a[1]+IAZ;
        mplsh(a,n);
    }
}

```

Finally, then, we arrive at a routine implementing equations (20.6.1) and (20.6.2):



```

3.1415926535897932384626433832795028841971693993751058209749445923078164062
862089986280348253421170679821480865132823066470938446095505822317253594081
284811174502841027019385211055596446229489549303819644288109756659334461284
756482337867831652712019091456485669234603486104543266482133936072602491412
737245870066063155881748815209209628292540917153643678925903600113305305488
204665213841469519415116094330572703657595919530921861173819326117931051185
480744623799627495673518857527248912279381830119491298336733624406566430860
213949463952247371907021798609437027705392171762931767523846748184676694051
320005681271452635608277857713427577896091736371787214684409012249534301465
495853710507922796892589235420199561121290219608640344181598136297747713099
605187072113499999983729780499510597317328160963185950244594553469083026425
223082533446850352619311881710100031378387528865875332083814206171776691473
035982534904287554687311595628638823537875937519577818577805321712268066130
019278766111959092164201989380952572010654858632788659361533818279682303019
520353018529689957736225994138912497217752834791315155748572424541506959508
295331168617278558890750983817546374649393192550604009277016711390098488240
128583616035637076601047101819429555961989467678374494482553797747268471040
475346462080466842590694912933136770289891521047521620569660240580381501935
112533824300355876402474964732639141992726042699227967823547816360093417216
412199245863150302861829745557067498385054945885869269956909272107975093029
553211653449872027559602364806654991198818347977535663698074265425278625518
184175746728909777272938000816470600161452491921732172147223501414419735685
481613611573525521334757418494684385233239073941433345477624168625189835694
855620992192221842725502542568876717904946016534668049886272327917860857843
838279679766814541009538837863609506800642251252051173929848960841284886269
45604241965285022106611863067442786220391949450471237137869609563643719172
874677646575739624138908658326459958133904780275900994657640789512694683983
525957098258226205224894077267194782684826014769909026401363944374553050682
034962524517493996514314298091906592509372216964615157098583874105978859597
729754989301617539284681382686838689427741559918559252459539594310499725246
808459872736446958486538367362226260991246080512438843904512441365497627807
977156914359977001296160894416948685558484063534220722258284886481584560285

```

Figure 20.6.1. The first 2398 decimal digits of  $\pi$ , computed by the routines in this section.

```

#include <stdio.h>
#include "nrutil.h"
#define IAOFF 48

void mppi(int n)
Demonstrate multiple precision routines by calculating and printing the first n bytes of  $\pi$ .
{
    void mp2dfr(unsigned char a[], unsigned char s[], int n, int *m);
    void mpadd(unsigned char w[], unsigned char u[], unsigned char v[], int n);
    void mpinv(unsigned char u[], unsigned char v[], int n, int m);
    void mplsh(unsigned char u[], int n);
    void mpmov(unsigned char u[], unsigned char v[], int n);
    void mpmul(unsigned char w[], unsigned char u[], unsigned char v[], int n,
               int m);
    void mpsdv(unsigned char w[], unsigned char u[], int n, int iv, int *ir);
    void mpsqrt(unsigned char w[], unsigned char u[], unsigned char v[], int n,
               int m);
    int ir,j,m;
    unsigned char mm,*x,*y,*sx,*sxi,*t,*s,*pi;

    x=cvector(1,n+1);
    y=cvector(1,n<<1);
    sx=cvector(1,n);
    sxi=cvector(1,n);
    t=cvector(1,n<<1);
    s=cvector(1,3*n);
    pi=cvector(1,n+1);
    t[1]=2;
    Set T = 2.
    for (j=2;j<=n;j++) t[j]=0;

```

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```

mpsqrt(x,x,t,n,n);          Set  $X_0 = \sqrt{2}$ .
mpadd(pi,t,x,n);          Set  $\pi_0 = 2 + \sqrt{2}$ .
mplsh(pi,n);
mpsqrt(sx,sxi,x,n,n);     Set  $Y_0 = 2^{1/4}$ .
mpmov(y,sx,n);
for (;) {
  mpadd(x,sx,sxi,n);      Set  $X_{i+1} = (X_i^{1/2} + X_i^{-1/2})/2$ .
  mpsdv(x,&x[1],n,2,&ir);
  mpsqrt(sx,sxi,x,n,n);   Form the temporary  $T = Y_i X_{i+1}^{1/2} + X_{i+1}^{-1/2}$ .
  mpmul(t,y,sx,n,n);
  mpadd(&t[1],&t[1],sxi,n);
  x[1]++;                 Increment  $X_{i+1}$  and  $Y_i$  by 1.
  y[1]++;
  mpinv(s,y,n,n);        Set  $Y_{i+1} = T/(Y_i + 1)$ .
  mpmul(y,&t[2],s,n,n);
  mplsh(y,n);
  mpmul(t,x,s,n,n);      Form temporary  $T = (X_{i+1} + 1)/(Y_i + 1)$ .
  mm=t[2]-1;             If  $T = 1$  then we have converged.
  for (j=3;j<=n;j++) {
    if (t[j] != mm) break;
  }
  m=t[n+1]-mm;
  if (j <= n || m > 1 || m < -1) {
    mpmul(s,pi,&t[1],n,n);   Set  $\pi_{i+1} = T\pi_i$ .
    mpmov(pi,&s[1],n);
    continue;
  }
  printf("pi=\n");
  s[1]=pi[1]+IAOFF;
  s[2]='.';
  m=mm;
  mp2dfr(&pi[1],&s[2],n-1,&m);
  Convert to decimal for printing. NOTE: The conversion routine, for this demonstration
  only, is a slow ( $\propto N^2$ ) algorithm. Fast ( $\propto N \ln N$ ), more complicated, radix conversion
  algorithms do exist.
  s[m+3]=0;
  printf(" %64s\n",&s[1]);
  free_cvector(pi,1,n+1);
  free_cvector(s,1,3*n);
  free_cvector(t,1,n<<1);
  free_cvector(sxi,1,n);
  free_cvector(sx,1,n);
  free_cvector(y,1,n<<1);
  free_cvector(x,1,n+1);
  return;
}
}

```

Figure 20.6.1 gives the result, computed with  $n = 1000$ . As an exercise, you might enjoy checking the first hundred digits of the figure against the first 12 terms of Ramanujan's celebrated identity [3]

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4n)! (1103 + 26390n)}{(n! 396^n)^4} \quad (20.6.6)$$

using the above routines. You might also use the routines to verify that the number  $2^{512} + 1$  is not a prime, but has factors 2,424,833 and 7,455,602,825,647,884,208,337,395,736,200,454,918,783,366,342,657 (which are in fact prime; the remaining prime factor being about  $7.416 \times 10^{98}$ ) [4].

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