which is related to the gamma function by

\[ B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z + w)} \]  \hspace{1cm} (6.1.9)

hence

```c
#include <math.h>

float beta(float z, float w)
Returns the value of the beta function \( B(z, w) \).
{
    float gammln(float xx);
    return exp(gammln(z)+gammln(w)-gammln(z+w));
}
```

CITED REFERENCES AND FURTHER READING:


### 6.2 Incomplete Gamma Function, Error Function, Chi-Square Probability Function, Cumulative Poisson Function

The incomplete gamma function is defined by

\[ P(a, x) = \frac{\gamma(a, x)}{\Gamma(a)} = \frac{1}{\Gamma(a)} \int_0^x e^{-t} t^{a-1} dt \quad (a > 0) \]  \hspace{1cm} (6.2.1)

It has the limiting values

\[ P(a, 0) = 0 \quad \text{and} \quad P(a, \infty) = 1 \]  \hspace{1cm} (6.2.2)

The incomplete gamma function \( P(a, x) \) is monotonic and (for \( a \) greater than one or so) rises from “near-zero” to “near-unity” in a range of \( x \) centered on about \( a - 1 \), and of width about \( \sqrt{a} \) (see Figure 6.2.1).

The complement of \( P(a, x) \) is also confusingly called an incomplete gamma function,

\[ Q(a, x) = 1 - P(a, x) = \frac{\Gamma(a, x)}{\Gamma(a)} = \frac{1}{\Gamma(a)} \int_x^\infty e^{-t} t^{a-1} dt \quad (a > 0) \]  \hspace{1cm} (6.2.3)
6.2 Incomplete Gamma Function

The incomplete gamma function $P(a, x)$ for four values of $a$.

It has the limiting values

$$Q(a, 0) = 1 \quad \text{and} \quad Q(a, \infty) = 0 \quad (6.2.4)$$

The notations $P(a, x), \gamma(a, x)$, and $\Gamma(a, x)$ are standard; the notation $Q(a, x)$ is specific to this book.

There is a series development for $\gamma(a, x)$ as follows:

$$\gamma(a, x) = e^{-x} x^a \sum_{n=0}^{\infty} \frac{\Gamma(a)}{\Gamma(a + 1 + n)} x^n \quad (6.2.5)$$

One does not actually need to compute a new $\Gamma(a + 1 + n)$ for each $n$; one rather uses equation (6.1.3) and the previous coefficient.

A continued fraction development for $\Gamma(a, x)$ is

$$\Gamma(a, x) = e^{-x} x^a \left( \frac{1}{x + 1 - a - \frac{1 \cdot (1 - a)}{x + 3 - a - \frac{2 \cdot (2 - a)}{x + 5 - a - \ldots}} \right) \quad (x > 0) \quad (6.2.6)$$

It is computationally better to use the even part of (6.2.6), which converges twice as fast (see §5.2):

$$\Gamma(a, x) = e^{-x} x^a \left( \frac{1}{x + 1 - a - \frac{1 \cdot (1 - a)}{x + 3 - a - \frac{2 \cdot (2 - a)}{x + 5 - a - \ldots}} \right) \quad (x > 0) \quad (6.2.7)$$

It turns out that (6.2.5) converges rapidly for $x$ less than about $a + 1$, while (6.2.6) or (6.2.7) converges rapidly for $x$ greater than about $a + 1$. In these respective...
regimes each requires at most a few times $\sqrt{a}$ terms to converge, and this many only near $x = a$, where the incomplete gamma functions are varying most rapidly. Thus (6.2.5) and (6.2.7) together allow evaluation of the function for all positive $a$ and $x$. An extra dividend is that we never need compute a function value near zero by subtracting two nearly equal numbers. The higher-level functions that return $P(a,x)$ and $Q(a,x)$ are

```c
float gammpl(float a, float x)
Returns the incomplete gamma function $P(a,x)$.
{
    void gcf(float *gammcf, float a, float x, float *gln);
    void gser(float *gamser, float a, float x, float *gln);
    void nrerror(char error_text[]);
    float gamser, gammcf, gln;
    if (x < 0.0 || a <= 0.0) nrerror("Invalid arguments in routine gammpl");
    if (x < (a+1.0)) {
        gser(&gamser,a,x,&gln);
        return gamser;
    } else {
        Use the continued fraction representation.
        gcf(&gammcf,a,x,&gln);
        return 1.0-gammcf;
    }
}

float gammql(float a, float x)
Returns the incomplete gamma function $Q(a,x) \equiv 1-P(a,x)$.
{
    void gcf(float *gammcf, float a, float x, float *gln);
    void gser(float *gamser, float a, float x, float *gln);
    void nrerror(char error_text[]);
    float gamser, gammcf, gln;
    if (x < 0.0 || a <= 0.0) nrerror("Invalid arguments in routine gammql");
    if (x < (a+1.0)) {
        gser(&gamser,a,x,&gln);
        return 1.0-gamser;
    } else {
        Use the continued fraction representation.
        gcf(&gammcf,a,x,&gln);
        return gammcf;
    }
}
```

The argument $\ln\Gamma(a)$; the reason for this is so that it is available to you if you want to modify the above two procedures to give $\gamma(a,x)$ and $\Gamma(a,x)$, in addition to $P(a,x)$ and $Q(a,x)$ (cf. equations 6.2.1 and 6.2.3).

The functions $\text{gser}$ and $\text{gcf}$ which implement (6.2.5) and (6.2.7) are

```c
#include <math.h>
#define ITMAX 100
#define EPS 3.0e-7

void gser(float *gamser, float a, float x, float *gln)
Returns the incomplete gamma function $P(a,x)$ evaluated by its series representation as $\text{gamser}$. Also returns $\ln\Gamma(a)$ as $\text{gln}$.
{
    float gamln(float xx);
```
6.2 Incomplete Gamma Function

```c
#include <math.h>
#define ITMAX 100
#define EPS 3.0e-7
#define FPMIN 1.0e-30

void gcf(float *gammcf, float a, float x, float *gln)
{
    float gammln(float xx);
    void nrerror(char error_text[]);
    int i;
    float an,b,c,d,del,h;

    *gln=gammln(a);
    b=x+1.0-a;
    c=1.0/FPMIN;
    d=1.0/b;
    h=d;
    for (i=1;i<=ITMAX;i++) {
        an = -i*(i-a);
        b *= 2.0;
        d=an+d*b;
        if (fabs(d) < FPMIN) d=FPMIN;
        c=b+an/c;
        if (fabs(c) < FPMIN) c=FPMIN;
        d=1.0/d;
        del=d*c;
        h *= del;
        if (fabs(del-1.0) < EPS) break;
    }
    if (i > ITMAX) nrerror("a too large, ITMAX too small in gcf");
    *gammcf=exp(-x+a*log(x)-(*gln))*h;
}
```

Maximum allowed number of iterations.
Relative accuracy.
Number near the smallest representable floating-point number.

Returns the incomplete gamma function $Q(a,x)$ evaluated by its continued fraction representation as `gammcf`. Also returns $\ln \Gamma(a)$ as `gln`.

```c
void gcf(float *gammcf, float a, float x, float *gln)
{
    float gammln(float xx);
    void nrerror(char error_text[]);
    int i;
    float an,b,c,d,del,h;

    *gln=gammln(a);
    b=x+1.0-a;
    c=1.0/FPMIN;
    d=1.0/b;
    h=d;
    for (i=1;i<=ITMAX;i++) {
        an = -i*(i-a);
        b *= 2.0;
        d=an+d*b;
        if (fabs(d) < FPMIN) d=FPMIN;
        c=b+an/c;
        if (fabs(c) < FPMIN) c=FPMIN;
        d=1.0/d;
        del=d*c;
        h *= del;
        if (fabs(del-1.0) < EPS) break;
    }
    if (i > ITMAX) nrerror("a too large, ITMAX too small in gcf");
    *gammcf=exp(-x+a*log(x)-(*gln))*h;
}
```
Error Function

The error function and complementary error function are special cases of the incomplete gamma function, and are obtained moderately efficiently by the above procedures. Their definitions are

\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt
\]

and

\[
\text{erfc}(x) \equiv 1 - \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt
\]

The functions have the following limiting values and symmetries:

\[
\text{erf}(0) = 0 \quad \text{erf}(\infty) = 1 \quad \text{erf}(-x) = -\text{erf}(x)
\]

\[
\text{erfc}(0) = 1 \quad \text{erfc}(\infty) = 0 \quad \text{erfc}(-x) = 2 - \text{erfc}(x)
\]

They are related to the incomplete gamma functions by

\[
\text{erf}(x) = P\left(\frac{1}{2}, x^2\right) \quad (x \geq 0)
\]

and

\[
\text{erfc}(x) = Q\left(\frac{1}{2}, x^2\right) \quad (x \geq 0)
\]

We'll put an extra “f” into our routine names to avoid conflicts with names already in some C libraries:

```c
float erff(float x)
Returns the error function erf(x).
{
    float gammp(float a, float x);
    return x < 0.0 ? -gammp(0.5,x*x) : gammp(0.5,x*x);
}
```

```c
float erffc(float x)
Returns the complementary error function erfc(x).
{
    float gammp(float a, float x);
    float gammq(float a, float x);
    return x < 0.0 ? 1.0-gammp(0.5,x*x) : gammq(0.5,x*x);
}
```

If you care to do so, you can easily remedy the minor inefficiency in erff and erffc, namely that \(\Gamma(0.5) = \sqrt{\pi}\) is computed unnecessarily when gammp or gammq is called. Before you do that, however, you might wish to consider the following routine, based on Chebyshev fitting to an inspired guess as to the functional form:
#include <math.h>

float erfcc(float x)
Returns the complementary error function erfc(x) with fractional error everywhere less than $1.2 \times 10^{-7}$.
{
  float t,z,ans;
  z=fabs(x);
  t=1.0/(1.0+0.5*z);
  ans=t*exp(-z*z-1.26551223+t*(1.00002368+t*(-0.18628806+t*(-0.82215223+t*0.17087277)))));
  return x >= 0.0 ? ans : 2.0-ans;
}

There are also some functions of two variables that are special cases of the incomplete gamma function:

**Cumulative Poisson Probability Function**

$P_x(< k)$, for positive $x$ and integer $k \geq 1$, denotes the cumulative Poisson probability function. It is defined as the probability that the number of Poisson random events occurring will be between 0 and $k-1$ inclusive, if the expected mean number is $x$. It has the limiting values

$$P_x(< 1) = e^{-x} \quad P_x(< \infty) = 1 \quad (6.2.14)$$

Its relation to the incomplete gamma function is simply

$$P_x(< k) = Q(k,x) = \text{gammq}(k,x) \quad (6.2.15)$$

**Chi-Square Probability Function**

$P(\chi^2|\nu)$ is defined as the probability that the observed chi-square for a correct model should be less than a value $\chi^2$. (We will discuss the use of this function in Chapter 15.) Its complement $Q(\chi^2|\nu)$ is the probability that the observed chi-square will exceed the value $\chi^2$ by chance even for a correct model. In both cases $\nu$ is an integer, the number of degrees of freedom. The functions have the limiting values

$$P(0|\nu) = 0 \quad P(\infty|\nu) = 1 \quad (6.2.16)$$
$$Q(0|\nu) = 1 \quad Q(\infty|\nu) = 0 \quad (6.2.17)$$

and the following relation to the incomplete gamma functions,

$$P(\chi^2|\nu) = P \left( \frac{\nu}{2}, \frac{\chi^2}{2} \right) = \text{gamm} \left( \frac{\nu}{2}, \frac{\chi^2}{2} \right) \quad (6.2.18)$$
$$Q(\chi^2|\nu) = Q \left( \frac{\nu}{2}, \frac{\chi^2}{2} \right) = \text{gammq} \left( \frac{\nu}{2}, \frac{\chi^2}{2} \right) \quad (6.2.19)$$
6.3 Exponential Integrals

The standard definition of the exponential integral is

\[ E_n(x) = \int_1^\infty \frac{e^{-xt}}{t^n} \, dt, \quad x > 0, \quad n = 0, 1, \ldots \]  \hspace{1cm} (6.3.1)

The function defined by the principal value of the integral

\[ \text{Ei}(x) = -\int_{-\infty}^x \frac{e^t}{t} \, dt = \int_x^{\infty} \frac{e^{-t}}{t} \, dt, \quad x > 0 \]  \hspace{1cm} (6.3.2)

is also called an exponential integral. Note that \( \text{Ei}(-x) \) is related to \( -E_1(x) \) by analytic continuation.

The function \( E_n(x) \) is a special case of the incomplete gamma function

\[ E_n(x) = x^{n-1} \Gamma(1 - n, x) \]  \hspace{1cm} (6.3.3)

We can therefore use a similar strategy for evaluating it. The continued fraction — just equation (6.2.6) rewritten — converges for all \( x > 0 \):

\[ E_n(x) = e^{-x} \left( \frac{1}{x + 1} + \frac{n}{x + 1} + \frac{n+1}{x + 1} + \cdots \right) \]  \hspace{1cm} (6.3.4)

We use it in its more rapidly converging even form,

\[ E_n(x) = e^{-x} \left( \frac{1}{x + n} - \frac{1}{x + n + 2} + \frac{2(n+1)}{x + n + 4} - \cdots \right) \]  \hspace{1cm} (6.3.5)

The continued fraction only really converges fast enough to be useful for \( x \gtrsim 1 \). For \( 0 < x \lesssim 1 \), we can use the series representation

\[ E_n(x) = \frac{(-x)^{n-1}}{(n-1)!} \left[ -\ln x + \psi(n) \right] - \sum_{m=0}^{\infty} \frac{(-x)^m}{(m - n + 1)m!} \]  \hspace{1cm} (6.3.6)

The quantity \( \psi(n) \) here is the digamma function, given for integer arguments by

\[ \psi(1) = -\gamma, \quad \psi(n) = -\gamma + \sum_{m=1}^{n-1} \frac{1}{m} \]  \hspace{1cm} (6.3.7)