Chapter 12. Fast Fourier Transform

integer arithmetic modulo some large prime \( N+1 \), and the \( N \)th root of 1 by the modulo arithmetic equivalent. Strictly speaking, these are not Fourier transforms at all, but the properties are quite similar and computational speed can be far superior. On the other hand, their use is somewhat restricted to quantities like correlations and convolutions since the transform itself is not easily interpretable as a "frequency" spectrum.

CITED REFERENCES AND FURTHER READING:

12.3 FFT of Real Functions, Sine and Cosine Transforms

It happens frequently that the data whose FFT is desired consist of real-valued samples \( f_j \), \( j = 0 \ldots N - 1 \). To use four1, we put these into a complex array with all imaginary parts set to zero. The resulting transform \( F_n \), \( n = 0 \ldots N - 1 \) satisfies \( F_{N-n}^* = F_n \). Since this complex-valued array has real values for \( F_0 \) and \( F_{N/2} \), and \( (N/2) - 1 \) other independent values \( F_1 \ldots F_{N/2-1} \), it has the same \( 2(N/2 - 1) + 2 = N \) “degrees of freedom” as the original, real data set. However, the use of the full complex FFT algorithm for real data is inefficient, both in execution time and in storage required. You would think that there is a better way.

There are two better ways. The first is “mass production”: Pack two separate real functions into the input array in such a way that their individual transforms can be separated from the result. This is implemented in the program twofft below. This may remind you of a one-cent sale, at which you are coerced to purchase two of an item when you only need one. However, remember that for correlations and convolutions the Fourier transforms of two functions are involved, and this is a handy way to do them both at once. The second method is to pack the real input array cleverly, without extra zeros, into a complex array of half its length. One then performs a complex FFT on this shorter length; the trick is then to get the required answer out of the result. This is done in the program realft below.
Transform of Two Real Functions Simultaneously

First we show how to exploit the symmetry of the transform \( F_n \) to handle two real functions at once: Since the input data \( f_j \) are real, the components of the discrete Fourier transform satisfy

\[
F_{N-n} = (F_n)^* \tag{12.3.1}
\]

where the asterisk denotes complex conjugation. By the same token, the discrete Fourier transform of a purely imaginary set of \( g_j \)'s has the opposite symmetry.

\[
G_{N-n} = -(G_n)^* \tag{12.3.2}
\]

Therefore we can take the discrete Fourier transform of two real functions each of length \( N \) simultaneously by packing the two data arrays as the real and imaginary parts, respectively, of the complex input array of \( \text{four1} \). Then the resulting transform array can be unpacked into two complex arrays with the aid of the two symmetries. Routine \( \text{twofft} \) works out these ideas.

```fortran
SUBROUTINE twofft(data1,data2,fft1,fft2,n)
INTEGER n
REAL data1(n),data2(n)
COMPLEX fft1(n),fft2(n)
C USES four1
Given two real input arrays data1(1:n) and data2(1:n), this routine calls four1 and returns two complex output arrays, fft1(1:n) and fft2(1:n), each of complex length \( n \) (i.e., real length \( 2*n \)), which contain the discrete Fourier transforms of the respective data arrays. \( n \) MUST be an integer power of 2.
INTEGER j,n2
COMPLEX h1,h2,c1,c2
c1=cmplx(0.5,0.0)
c2=cmplx(0.0,-0.5)
do 11 j=1,n
   fft1(j)=cmplx(data1(j),data2(j))
   Pack the two real arrays into one complex array.
11 enddo
   call four1(fft1,n,1)
   Transform the complex array.
   fft2(1)=cmplx(aimag(fft1(1)),0.0)
   fft1(1)=cmplx(real(fft1(1)),0.0)
   n2=n+2
   do 12 j=2,n/2+1
      h1=c1*(fft1(j)+conjg(fft1(n2-j)))
      Use symmetries to separate the two transforms.
      h2=c2*(fft1(j)-conjg(fft1(n2-j)))
      Ship them out in two complex arrays.
      fft1(j)=h1
      fft1(n2-j)=conjg(h1)
      fft2(j)=h2
      fft2(n2-j)=conjg(h2)
12 enddo
   return
END
```

What about the reverse process? Suppose you have two complex transform arrays, each of which has the symmetry (12.3.1), so that you know that the inverses of both transforms are real functions. Can you invert both in a single FFT? This is even easier than the other direction. Use the fact that the FFT is linear and form the sum of the first transform plus \( i \) times the second. Invert using \( \text{four1} \) with
isign = −1. The real and imaginary parts of the resulting complex array are the two desired real functions.

### FFT of Single Real Function

To implement the second method, which allows us to perform the FFT of a single real function without redundancy, we split the data set in half, thereby forming two real arrays of half the size. We can apply the program above to these two, but of course the result will not be the transform of the original data. It will be a schizophrenic combination of two transforms, each of which has half of the information we need. Fortunately, this schizophrenia is treatable. It works like this:

The right way to split the original data is to take the even-numbered \( f_j \) as one data set, and the odd-numbered \( f_j \) as the other. The beauty of this is that we can take the original real array and treat it as a complex array \( h_j \) of half the length. The first data set is the real part of this array, and the second is the imaginary part, as prescribed for `twofft`. No repacking is required. In other words

\[
\begin{align*}
    h_{j} &= f_{2j} + i f_{2j+1}, \quad j = 0, \ldots, N/2 - 1. \\
    \text{We submit this to `four1`, and it will return a complex array}
\end{align*}
\]

\[
\begin{align*}
    H_{n} &= F_{n}^e + i F_{n}^o, \quad n = 0, \ldots, N/2 - 1
\end{align*}
\]

with

\[
\begin{align*}
    F_{n}^e &= \sum_{k=0}^{N/2-1} f_{2k} e^{2\pi i kn/(N/2)} \\
    F_{n}^o &= \sum_{k=0}^{N/2-1} f_{2k+1} e^{2\pi i kn/(N/2)}
\end{align*}
\]

(12.3.3)

The discussion of program `twofft` tells you how to separate the two transforms \( F_{n}^e \) and \( F_{n}^o \) out of \( H_{n} \). How do you work them into the transform \( F_{n} \) of the original data set \( f_j \)? Simply glance back at equation (12.2.3):

\[
F_{n} = F_{n}^e + e^{2\pi i n/N} F_{n}^o \quad n = 0, \ldots, N - 1
\]

(12.3.4)

Expressed directly in terms of the transform \( H_{n} \) of our real (masquerading as complex) data set, the result is

\[
F_{n} = \frac{1}{2} (H_{n} + H_{N/2-n}^*) - \frac{i}{2} (H_{n} - H_{N/2-n}^*) e^{2\pi i n/N} \quad n = 0, \ldots, N - 1
\]

(12.3.5)

A few remarks:

- Since \( F_{N-n}^* = F_{n} \) there is no point in saving the entire spectrum. The positive frequency half is sufficient and can be stored in the same array as the original data. The operation can, in fact, be done in place.
- Even so, we need values \( H_{n} \), \( n = 0, \ldots, N/2 \) whereas `four1` returns only the values \( n = 0, \ldots, N/2 - 1 \). Symmetry to the rescue, \( H_{N/2} = H_{0} \).
- The values \( F_{0} \) and \( F_{N/2} \) are real and independent. In order to actually get the entire \( F_{n} \) in the original array space, it is convenient to return \( F_{N/2} \) as the imaginary part of \( F_{0} \).
• Despite its complicated form, the process above is invertible. First peel \( F_{N/2} \) out of \( F_0 \). Then construct

\[
F_n^e = \frac{1}{2} \left( F_n + F_{N/2-n}^* \right) \quad n = 0, \ldots, N/2 - 1
\]

\[
F_n^o = \frac{1}{2} e^{-2\pi in/N} \left( F_n - F_{N/2-n}^* \right)
\]

and use four1 to find the inverse transform of \( H_n = F_n^{(1)} + iF_n^{(2)} \). Surprisingly, the actual algebraic steps are virtually identical to those of the forward transform.

Here is a representation of what we have said:

```fortran
SUBROUTINE realft(data,n,isign)
INTEGER isign,n
REAL data(n)
C USES four1
Calculates the Fourier transform of a set of \( n \) real-valued data points. Replaces this data (which is stored in array data(1:n)) by the positive frequency half of its complex Fourier transform. The real-valued first and last components of the complex transform are returned as elements data(1) and data(2), respectively. \( n \) must be a power of 2. This routine also calculates the inverse transform of a complex data array if it is the transform of real data. (Result in this case must be multiplied by \( 2/n \).)
INTEGER i,i1,i2,i3,i4,n2p3
REAL c1,c2,h1r,h1i,h2r,h2i,wi,wri,wpr
DOUBLE PRECISION theta,wi,wpi,wtemp
Double precision for the trigonometric recurrences.
theta=3.141592653589793d0/dble(n/2)
Initialize the recurrence.
c1=0.5
if (isign.eq.1) then
  c2=-0.5
  call four1(data,n/2,+1)
The forward transform is here.
else
  c2=0.5
  theta=-theta
endif
wpr=-2.0d0*sin(0.5d0*theta)**2
wpi=sin(theta)
wri=1.0d0+wpr
wi=wpi
n2p3=n+3
do 11 i=2,n/4
  Case i=1 done separately below.
i1=2*i-1
i2=i1+1
i3=n2p3-i2
i4=i3+1
wri=sngl(wri)
  The two separate transforms are separated out of data.
wi=sngl(wi)
h1r=c1*(data(i1)+data(i3))
h1i=c1*(data(i2)-data(i4))
h2r=c2*(data(i2)+data(i4))
h2i=c2*(data(i1)-data(i3))
data(i1)=h1r+wri*h2r-wi*h2i
  Here they are recombined to form the true transform of the original real data.
data(i2)=h1i+wri*h2i+wi*h2r
data(i3)=h1r-wri*h2r+wi*h2i
data(i4)=-h1i+wri*h2i+wi*h2r
wtemp=wri
  The recurrence.
wri=wri*wpr+wi*wri+wri
wi=wi*wpr+wtemp*wpi+wi
enddo 11
```
if (isign.eq.1) then
    h1r=data(1)
    data(1)=h1r+data(2)
    data(2)=h1r-data(2)
else
    h1r=data(1)
    data(1)=c1*(h1r+data(2))
    data(2)=c1*(h1r-data(2))
    call four1(data,n/2,-1)
endif
return
END

Fast Sine and Cosine Transforms

Among their other uses, the Fourier transforms of functions can be used to solve differential equations (see §19.4). The most common boundary conditions for the solutions are 1) they have the value zero at the boundaries, or 2) their derivatives are zero at the boundaries. In the first instance, the natural transform to use is the sine transform, given by

$$F_k = \sum_{j=1}^{N-1} f_j \sin(\pi jk/N)$$

where $f_j$, $j = 0, \ldots, N-1$ is the data array, and $f_0 \equiv 0$.

At first blush this appears to be simply the imaginary part of the discrete Fourier transform. However, the argument of the sine differs by a factor of two from the value that would make this so. The sine transform uses sines only as a complete set of functions in the interval from 0 to $2\pi$, and, as we shall see, the cosine transform uses cosines only. By contrast, the normal FFT uses both sines and cosines, but only half as many of each. (See Figure 12.3.1.)

The expression (12.3.7) can be "force-fit" into a form that allows its calculation via the FFT. The idea is to extend the given function rightward past its last tabulated value. We extend the data to twice their length in such a way as to make them an odd function about $j = N$, with $f_N = 0$,

$$f_{2N-j} = -f_j \quad j = 0, \ldots, N-1$$

Consider the FFT of this extended function:

$$F_k = \sum_{j=0}^{2N-1} f_j e^{2\pi ijk/(2N)}$$

The half of this sum from $j = N$ to $j = 2N-1$ can be rewritten with the substitution $j' = 2N - j$,

$$\sum_{j=N}^{2N-1} f_j e^{2\pi ijk/(2N)} = \sum_{j'=1}^{N} f_{2N-j'} e^{2\pi i(2N-j')k/(2N)}$$

$$= - \sum_{j'=0}^{N-1} f_{j'} e^{-2\pi ijk/(2N)}$$

(12.3.10)
so that

\[
F_k = \sum_{j=0}^{N-1} f_j \left[ e^{2\pi ijk/(2N)} - e^{-2\pi ijk/(2N)} \right]
\]

(12.3.11)

Thus, up to a factor $2i$ we get the sine transform from the FFT of the extended function.

This method introduces a factor of two inefficiency into the computation by extending the data. This inefficiency shows up in the FFT output, which has zeros for the real part of every element of the transform. For a one-dimensional problem, the factor of two may be bearable, especially in view of the simplicity of the method. When we work with partial differential equations in two or three dimensions, though, the factor becomes four or eight, so efforts to eliminate the inefficiency are well rewarded.
From the original real data array \( f_j \) we will construct an auxiliary array \( y_j \) and apply to it the routine \texttt{realft}. The output will then be used to construct the desired transform. For the sine transform of data \( f_j, j = 1, \ldots, N - 1 \), the auxiliary array is

\[
y_0 = 0
\]

\[
y_j = \sin(j\pi/N)(f_j + f_{N-j}) + \frac{1}{2}(f_j - f_{N-j}) \quad j = 1, \ldots, N - 1
\]  \hspace{1cm} (12.3.12)

This array is of the same dimension as the original. Notice that the first term is symmetric about \( j = N/2 \) and the second is antisymmetric. Consequently, when \texttt{realft} is applied to \( y_j \), the result has real parts \( R_k \) and imaginary parts \( I_k \) given by

\[
R_k = \sum_{j=0}^{N-1} y_j \cos(2\pi j k / N)
\]

\[
= \sum_{j=1}^{N-1} (f_j + f_{N-j}) \sin(j\pi/N) \cos(2\pi j k / N)
\]

\[
= \sum_{j=0}^{N-1} 2f_j \sin(j\pi/N) \cos(2\pi j k / N)
\]

\[
= \sum_{j=0}^{N-1} f_j \left[ \sin\left(\frac{(2k+1)j\pi}{N}\right) - \sin\left(\frac{(2k-1)j\pi}{N}\right) \right]
\]

\[
= F_{2k+1} - F_{2k-1}
\]  \hspace{1cm} (12.3.13)

\[
I_k = \sum_{j=0}^{N-1} y_j \sin(2\pi j k / N)
\]

\[
= \sum_{j=1}^{N-1} (f_j - f_{N-j}) \frac{1}{2} \sin(2\pi j k / N)
\]

\[
= \sum_{j=0}^{N-1} f_j \sin(2\pi j k / N)
\]

\[
= F_{2k}
\]  \hspace{1cm} (12.3.14)

Therefore \( F_k \) can be determined as follows:

\[
F_{2k} = I_k \quad F_{2k+1} = F_{2k-1} + R_k \quad k = 0, \ldots, (N/2 - 1)
\]  \hspace{1cm} (12.3.15)

The even terms of \( F_k \) are thus determined very directly. The odd terms require a recursion, the starting point of which follows from setting \( k = 0 \) in equation (12.3.15) and using \( F_1 = -F_{-1} \):

\[
F_1 = \frac{1}{2} R_0
\]  \hspace{1cm} (12.3.16)

The implementing program is
The sine transform, curiously, is its own inverse. If you apply it twice, you get the original data, but multiplied by a factor of $N/2$.

The other common boundary condition for differential equations is that the derivative of the function is zero at the boundary. In this case the natural transform is the *cosine* transform. There are several possible ways of defining the transform. Each can be thought of as resulting from a different way of extending a given array to create an even array of double the length, and/or from whether the extended array contains $2N-1$, $2N$, or some other number of points. In practice, only two of the numerous possibilities are useful so we will restrict ourselves to just these two.

The first form of the cosine transform uses $N+1$ data points:

$$F_k = \frac{1}{2} \left[ f_0 + (-1)^k f_N \right] + \sum_{j=1}^{N-1} f_j \cos(\pi jk/N) \quad (12.3.17)$$

It results from extending the given array to an even array about $j = N$, with

$$f_{2N-j} = f_j, \quad j = 0, \ldots, N-1 \quad (12.3.18)$$
If you substitute this extended array into equation (12.3.9), and follow steps analogous to those leading up to equation (12.3.11), you will find that the Fourier transform is just twice the cosine transform (12.3.17). Another way of thinking about the formula (12.3.17) is to notice that it is the Chebyshev Gauss-Lobatto quadrature formula (see §4.5), often used in Clenshaw-Curtis adaptive quadrature (see §5.9, equation 5.9.4).

Once again the transform can be computed without the factor of two inefficiency. In this case the auxiliary function is

\[ y_j = \frac{1}{2}(f_j + f_{N-j}) - \sin(j\pi/N)(f_j - f_{N-j}) \quad j = 0, \ldots, N - 1 \]  

(12.3.19)

Instead of equation (12.3.15), realft now gives

\[ F_{2k} = R_k \quad F_{2k+1} = F_{2k-1} + I_k \quad k = 0, \ldots, (N/2 - 1) \]  

(12.3.20)

The starting value for the recursion for odd \( k \) in this case is

\[ F_1 = \frac{1}{2}(f_0 - f_N) + \sum_{j=1}^{N-1} f_j \cos(j\pi/N) \]  

(12.3.21)

This sum does not appear naturally among the \( R_k \) and \( I_k \), and so we accumulate it during the generation of the array \( y_j \).

Once again this transform is its own inverse, and so the following routine works for both directions of the transformation. Note that although this form of the cosine transform has \( N + 1 \) input and output values, it passes an array only of length \( N \) to realft.

```
SUBROUTINE cosft1(y,n)
INTEGER n
REAL y(n+1)
C USES realft
Calculates the cosine transform of a set y(1:n+1) of real-valued data points. The transformed data replace the original data in array y. n must be a power of 2. This program, without changes, also calculates the inverse cosine transform, but in this case the output array should be multiplied by 2/n.
INTEGER j
REAL sum,y1,y2
DOUBLE PRECISION theta,wi,wpi,wpr,wr,wtemp
For trig. recurrences.
theta=3.141592653589793d0/n
Initialize the recurrence.
wr=1.0d0
wi=0.0d0
wpr=-2.0d0*sin(0.5d0*theta)**2
wpi=sin(theta)
sum=0.5*(y(1)-y(n+1))
y(1)=0.5*(y(1)+y(n+1))
do 11 j=1,n/2-1
   y1=0.5*(y(j+1)+y(n-j+1))
   y2=(y(j+1)-y(n-j+1))
   y(j+1)=y1-wi*y2
   y(n-j+1)=y1+wi*y2
   sum=sum+wr*y2
   enddo
11 sum=sum+wr*y2
end
```

This program is designed for use in programs which compute large numbers of Fourier transforms, where the cost of computing the sine and cosine terms is negligible compared to the cost of the recursion. The program is given in full (including all the subroutines realft and cosft1) on the leaf of the front cover of this book.
The second important form of the cosine transform is defined by

$$F_k = \sum_{j=0}^{N-1} f_j \cos \frac{\pi k (j + \frac{1}{2})}{N}$$

(12.3.22)

with inverse

$$f_j = \frac{2}{N} \sum_{k=0}^{N-1} F_k \cos \frac{\pi k (j + \frac{1}{2})}{N}$$

(12.3.23)

Here the prime on the summation symbol means that the term for $k = 0$ has a coefficient of $\frac{1}{2}$ in front. This form arises by extending the given data, defined for $j = 0, \ldots, N-1$, to $j = N, \ldots, 2N-1$ in such a way that it is even about the point $N - \frac{1}{2}$ and periodic. (It is therefore also even about $j = -\frac{1}{2}$.) The form (12.3.23) is related to Gauss-Chebyshev quadrature (see equation 4.5.19), to Chebyshev approximation (§5.8, equation 5.8.7), and Clenshaw-Curtis quadrature (§5.9).

This form of the cosine transform is useful when solving differential equations on "staggered" grids, where the variables are centered midway between mesh points. It is also the standard form in the field of data compression and image processing.

The auxiliary function used in this case is similar to equation (12.3.19):

$$y_j = \frac{1}{2} (f_j + f_{N-j-1}) - \sin \frac{\pi (j + \frac{1}{2})}{N} (f_j - f_{N-j-1}) \quad j = 0, \ldots, N-1$$

(12.3.24)

Carrying out the steps similar to those used to get from (12.3.12) to (12.3.15), we find

$$F_{2k} = \cos \frac{\pi k}{N} R_k - \sin \frac{\pi k}{N} I_k$$

(12.3.25)

$$F_{2k-1} = \sin \frac{\pi k}{N} R_k + \cos \frac{\pi k}{N} I_k + F_{2k+1}$$

(12.3.26)

Note that equation (12.3.26) gives

$$F_{N-1} = \frac{1}{2} R_{N/2}$$

(12.3.27)

Thus the even components are found directly from (12.3.25), while the odd components are found by recursing (12.3.26) down from $k = N/2 - 1$, using (12.3.27) to start.

Since the transform is not self-inverting, we have to reverse the above steps to find the inverse. Here is the routine:
SUBROUTINE cosft2(y,n,isign)
INTEGER isign,n
REAL y(n)
C USES realft
Calculates the "staggered" cosine transform of a set y(1:n) of real-valued data points.
The transformed data replace the original data in array y. n must be a power of 2. Set
isign to +1 for a transform, and to −1 for an inverse transform. For an inverse transform,
the output array should be multiplied by 2/n.
INTEGER i
REAL sum,sum1,y1,y2,ytemp
DOUBLE PRECISION theta,wi,wi1,wpi,wr,wrl,wtemp,PI
Double precision for the trigonometric recurrences.
PARAMETER (PI=3.141592653589793d0)
theta=0.5d0*PI/n
Initialize the recurrences.
wr=1.0d0
wi=0.0d0
wr=cos(theta)
wi=sin(theta)
wpr=-2.0d0*wi1**2
wpi=sin(2.d0*theta)

if(isign.eq.1)then
  Forward transform.
    do 11 i=1,n/2
      y1=0.5*(y(i)+y(n-i+1))
      Calculate the auxiliary function.
      y2=wi1*(y(i)-y(n-i+1))
      y(i)=y1+y2
      y(n-i+1)=y1-y2
      wtemp=wr1
      Carry out the recurrence.
      wr1=wr1*wpr-wi1*wpi+wr1
      wi1=wi1*wpr+wtemp*wpi+wi1
    enddo 11
  call realft(y,n,1)
  Calculate the transform of the auxiliary function.
  do 12 i=3,n,2
    wtemp=wr
    Even terms.
    wr=wr*wpr-wi*wpi+wr
    wi=wi*wpr+wtemp*wpi+wi
    y1=y(i)+y(i+1)*wi
    y2=y(i)+y(i+1)*wi
    y(i)=y1
    y(i+1)=y2
    enddo 12
  sum=0.5*y(2)
  Initialize recurrence for odd terms with \( \frac{1}{2} R_{N/2} \).
  do 13 i=n,2,-2
    sum1=sum
    sum=sum+y(i)
    y(i)=sum1
  enddo 13
else if(isign.eq.-1)then
  Inverse transform.
    ytemp=y(n)
    Form difference of odd terms.
    do 14 i=n,4,-2
      y(i)=y(i+2)-y(i)
    enddo 14
    y(2)=2.0*ytemp
    Calculate \( R_k \) and \( I_k \).
    do 15 i=3,n,2
      wtemp=wr
      do 16 i=1,n/2
        y1=y(i)+y(n-i+1)
        Invert auxiliary array.
        wtemp=wr
        wi=wi*wpr+wtemp*wpi+wi
        y1=y(i)+y(i+1)*wi
        y2=y(i)+y(i+1)*wi
        y(i)=y1
        y(i+1)=y2
      enddo 15
    enddo 16
  call realft(y,n,-1)
An alternative way of implementing this algorithm is to form an auxiliary function by copying the even elements of \( f_j \) into the first \( N/2 \) locations, and the odd elements into the next \( N/2 \) elements in reverse order. However, it is not easy to implement the alternative algorithm without a temporary storage array and we prefer the above in-place algorithm.

Finally, we mention that there exist fast cosine transforms for small \( N \) that do not rely on an auxiliary function or use an FFT routine. Instead, they carry out the transform directly, often coded in hardware for fixed \( N \) of small dimension [1].

CITED REFERENCES AND FURTHER READING:

### 12.4 FFT in Two or More Dimensions

Given a complex function \( h(k_1, k_2) \) defined over the two-dimensional grid \( 0 \leq k_1 \leq N_1 - 1, \ 0 \leq k_2 \leq N_2 - 1 \), we can define its two-dimensional discrete Fourier transform as a complex function \( H(n_1, n_2) \), defined over the same grid,

\[
H(n_1, n_2) \equiv \sum_{k_2=0}^{N_2-1} \sum_{k_1=0}^{N_1-1} \exp(2\pi ik_2n_2/N_2) \exp(2\pi ik_1n_1/N_1) h(k_1, k_2)
\]

(12.4.1)

By pulling the “subscripts 2” exponential outside of the sum over \( k_1 \), or by reversing the order of summation and pulling the “subscripts 1” outside of the sum over \( k_2 \),