Chi-Square Probability Function

\( P(\chi^2|\nu) \) is defined as the probability that the observed chi-square for a correct model should be less than a value \( \chi^2 \). (We will discuss the use of this function in Chapter 15.) Its complement \( Q(\chi^2|\nu) \) is the probability that the observed chi-square will exceed the value \( \chi^2 \) by chance even for a correct model. In both cases \( \nu \) is an integer, the number of degrees of freedom. The functions have the limiting values

\[
\begin{align*}
P(0|\nu) &= 0 & P(\infty|\nu) &= 1 \quad (6.2.16) \\
Q(0|\nu) &= 1 & Q(\infty|\nu) &= 0 \quad (6.2.17)
\end{align*}
\]

and the following relation to the incomplete gamma functions,

\[
\begin{align*}
P(\chi^2|\nu) &= P\left(\nu, \frac{\chi^2}{2}\right) = \text{gammp}\left(\nu, \frac{\chi^2}{2}\right) \quad (6.2.18) \\
Q(\chi^2|\nu) &= Q\left(\nu, \frac{\chi^2}{2}\right) = \text{gammq}\left(\nu, \frac{\chi^2}{2}\right) \quad (6.2.19)
\end{align*}
\]

CITED REFERENCES AND FURTHER READING:

6.3 Exponential Integrals

The standard definition of the exponential integral is

\[
E_n(x) = \int_1^\infty \frac{e^{-xt}}{t^n} \, dt, \quad x > 0, \quad n = 0, 1, \ldots \quad (6.3.1)
\]

The function defined by the principal value of the integral

\[
\text{Ei}(x) = -\int_{-x}^{\infty} \frac{e^{-t}}{t} \, dt = \int_{-\infty}^{x} \frac{e^t}{t} \, dt, \quad x > 0 \quad (6.3.2)
\]

is also called an exponential integral. Note that \( \text{Ei}(-x) \) is related to \( -E_1(x) \) by analytic continuation.

The function \( E_n(x) \) is a special case of the incomplete gamma function

\[
E_n(x) = x^{n-1} \Gamma(1-n, x) \quad (6.3.3)
\]
We can therefore use a similar strategy for evaluating it. The continued fraction — just equation (6.2.6) rewritten — converges for all $x > 0$:

$$E_n(x) = e^{-x} \left( \frac{1}{x + \frac{n}{1 + \frac{1}{x + \frac{n+1}{1 + \frac{2}{x + \ldots}}}} \right)$$ (6.3.4)

We use it in its more rapidly converging even form,

$$E_n(x) = e^{-x} \left( \frac{1}{x + n - \frac{1 \cdot n}{x + n + 2 - \frac{2(n + 1)}{x + n + 4 - \ldots}}} \right)$$ (6.3.5)

The continued fraction only really converges fast enough to be useful for $x > \sim 1$. For $0 < x \sim 1$, we can use the series representation

$$E_n(x) = (\frac{-x}{n-1}) \ln x + \psi(n) - \sum_{m=0}^{\infty} (-x)^m (m-n+1)/((m-n+1)m!$$ (6.3.6)

The quantity $\psi(n)$ here is the digamma function, given for integer arguments by

$$\psi(1) = -\gamma, \quad \psi(n) = -\gamma + \sum_{m=1}^{n-1} \frac{1}{m}$$ (6.3.7)

where $\gamma = 0.5772156649\ldots$ is Euler’s constant. We evaluate the expression (6.3.6) in order of ascending powers of $x$:

$$E_n(x) = - \left[ \frac{1}{(1-n)} + \frac{x}{(2-n \cdot 1)} + \frac{x^2}{(3-n \cdot 2)} - \ldots + \frac{(-x)^{n-2}}{(1)(n-2)!} \right]$$

$$+ \frac{(-x)^{n-1}}{(n-1)!} \ln x + \psi(n) - \left[ \frac{(-x)^n}{1 \cdot n!} + \frac{(-x)^{n+1}}{2 \cdot (n+1)!} + \ldots \right]$$ (6.3.8)

The first square bracket is omitted when $n = 1$. This method of evaluation has the advantage that for large $n$ the series converges before reaching the term containing $\psi(n)$. Accordingly, one needs an algorithm for evaluating $\psi(n)$ only for small $n$, $n \lesssim 20-40$. We use equation (6.3.7), although a table look-up would improve efficiency slightly.

Amos[1] presents a careful discussion of the truncation error in evaluating equation (6.3.8), and gives a fairly elaborate termination criterion. We have found that simply stopping when the last term added is smaller than the required tolerance works about as well.

Two special cases have to be handled separately:

$$E_0(x) = e^{-x}/x$$

$$E_n(0) = \frac{1}{n-1}, \quad n > 1$$ (6.3.9)
The routine \texttt{expint} allows fast evaluation of $E_n(x)$ to any accuracy \texttt{EPS} within the reach of your machine's word length for floating-point numbers. The only modification required for increased accuracy is to supply Euler's constant with enough significant digits. 

\begin{verbatim}
FUNCTION expint(n,x)
INTEGER n,MAXIT
REAL expint,x,EPS,FPMIN,EULER
PARAMETER (MAXIT=100,EPS=1.e-7,FPMIN=1.e-30,EULER=.5772156649)

Evaluates the exponential integral $E_n(x)$.

Parameters: MAXIT is the maximum allowed number of iterations; EPS is the desired relative error, not smaller than the machine precision; FPMIN is a number near the smallest representable floating-point number; EULER is Euler's constant $\gamma$.

INTEGER i,ii,nm1
REAL a,b,c,d,del,fact,h,psi

\textbf{CASES:}

\textit{Special case.}

if(nm1.eq.0)then
expint=-log(x)-EULER
endif

\textit{Evaluate series.}

if(nm1.ne.0)then
expint=-log(x)+psi
else
psi=EULER
\textbf{CASES:}

\textbf{Lentz's algorithm (§5.2).}

b=x+n

c=1./FPMIN

d=1./b

\textit{Denominators cannot be zero.}

h=d
\textbf{DO 11 i=1,MAXIT}

a=-i*(nm1+i)

b=b+2.

d=1./(a*d+b)

c=b+a/c

del=c*d

h=h*del

if(abs(del-1.).lt.EPS)then
expint=h*exp(-x)
return
endif

\textbf{CONTINUED FRACTION FAILED IN EXPINT}
\textbf{RETURN}
endif

enddo

\textit{Evaluate series.}

\textit{Set first term.}

if(nm1.ne.0)then
expint=1./nm1
else
expint=exp(-x)/x
endif

\textbf{DO 13 i=1,MAXIT}

fact=-fact*x/i

if(i.ne.nm1)then

\textit{Compute $\psi(n)$.}

psi=EULER

\textbf{DO 12 ii=1,nm1}

psi=psi+1./ii

\textbf{ENDO}

del=fact*(-log(x)+psi)
\textbf{ENDO}

else

psi=-EULER

\textbf{DO 12 ii=1,nm1}

psi=psi+1./ii

\textbf{ENDO}

del=fact*(-log(x)+psi)
\textbf{ENDO}

expint=expint+del

if(abs(del).lt.abs(expint)*EPS) return
\textbf{ENDO}
\end{verbatim}
A good algorithm for evaluating \( E_i \) is to use the power series for small \( x \) and the asymptotic series for large \( x \). The power series is

\[
E_i(x) = \gamma + \ln x + \frac{x}{1 \cdot 1!} + \frac{x^2}{2 \cdot 2!} + \cdots \tag{6.3.10}
\]

where \( \gamma \) is Euler’s constant. The asymptotic expansion is

\[
E_i(x) \sim \frac{e^x}{x} \left( 1 + \frac{1!}{x} + \frac{2!}{x^2} + \cdots \right) \tag{6.3.11}
\]

The lower limit for the use of the asymptotic expansion is approximately \( |\ln \text{EPS}| \), where \( \text{EPS} \) is the required relative error.

```fortran
FUNCTION ei(x)
INTEGER MAXIT
REAL ei,x,eps,euler,fpmin
PARAMETER (EPS=6.e-8,EULER=.57721566,MAXIT=100,FPMIN=1.e-30)
Computes the exponential integral \( E_i(x) \) for \( x > 0 \).
Parameters: EPS is the relative error, or absolute error near the zero of \( E_i \) at \( x = 0.3725 \);
EULER is Euler's constant \( \gamma \); MAXIT is the maximum number of iterations allowed; FPMIN
is a number near the smallest representable floating-point number.
INTEGER k
REAL fact,prev,sum,term
if(x.le.0.) pause 'bad argument in ei'
if(x.lt.FPMIN)then
  Special case: avoid failure of convergence test because of underflow.
  ei=log(x)+EULER
else if(x.le.-log(EPS))then
  Use power series.
  sum=0.
  fact=1.
  do 11 k=1,MAXIT
    fact=fact*x/k
    term=fact/k
    sum=sum+term
    if(term.lt.EPS*sum)goto 1
  enddo
  pause 'series failed in ei'
  1 ei=sum+log(x)+EULER
else
  Use asymptotic series.
  Start with second term.
  sum=0.
  term=1.
  do 12 k=1,MAXIT
    prev=term
    term=term*k/x
    if(term.lt.EPS*sum)goto 2
    if(term.lt.prev)then
      sum=sum+term
    else
      sum=sum-prev
    endif
  enddo
  2 ei=exp(x)*(1.+sum)/x
endif
```
6.4 Incomplete Beta Function, Student’s Distribution, F-Distribution, Cumulative Binomial Distribution

The incomplete beta function is defined by

\[ I_x(a, b) = \frac{B_x(a, b)}{B(a, b)} = \frac{1}{B(a, b)} \int_0^x t^{a-1}(1-t)^{b-1} dt \quad (a, b > 0) \]  \tag{6.4.1}

It has the limiting values

\[ I_0(a, b) = 0 \quad I_1(a, b) = 1 \]  \tag{6.4.2}

and the symmetry relation

\[ I_x(a, b) = 1 - I_{1-x}(b, a) \]  \tag{6.4.3}

If \( a \) and \( b \) are both rather greater than one, then \( I_x(a, b) \) rises from “near-zero” to “near-unity” quite sharply at about \( x = a/(a+b) \). Figure 6.4.1 plots the function for several pairs \((a, b)\).

The incomplete beta function has a series expansion

\[ I_x(a, b) = \frac{x^a(1-x)^b}{aB(a, b)} \left[ 1 + \sum_{n=0}^{\infty} \frac{B(a+1, n+1)}{B(a+b, n+1)} x^{n+1} \right] , \]  \tag{6.4.4}

but this does not prove to be very useful in its numerical evaluation. (Note, however, that the beta functions in the coefficients can be evaluated for each value of \( n \) with just the previous value and a few multiplies, using equations 6.1.9 and 6.1.3.)

The continued fraction representation proves to be much more useful,

\[ I_x(a, b) = \frac{x^a(1-x)^b}{aB(a, b)} \left[ \frac{1}{1 + \frac{d_1}{1 + \frac{d_2}{1 + \cdots}} \right] \]  \tag{6.4.5}