The Characteristic Length Scale of the Magnetic Fluctuation in a Sunspot Penumbra: A Stochastic Polarized Radiative Transfer Approach

T. A. Carroll and J. Staude
Astrophysikalisches Institut Potsdam, An der Sternwarte 16, D-14482 Potsdam, Germany

Abstract. The characteristic size of penumbral structures are still below the current resolution limit of modern solar telescopes. Though we have seen a significant progress in theoretical work over the last decades no tight constraints can be placed on the size of penumbral structures in order to favor models with relatively large and thick magnetic flux elements, just at or below the current resolution limit, or on the other hand, clusters of optically thin micro-structures. Based on a macroscopic 2-component inversion and the approach of polarized radiative transfer in stochastic media, we have estimated the characteristic length scale of the magnetic fluctuation in a sunspot penumbra from observed Stokes spectra. The results yield a coherent picture for the entire magnetic neutral line of the penumbra and indicate that the magnetic fluctuations have a typical length scale between 30 km and 70 km.

1. Introduction

Solar sunspots are the most prominent manifestation of concentrated magnetic fields on the solar surface and have been the subject of an exhaustive amount of research. Despite the fact that the photospheric structure and morphology is well studied and described, and despite the advances in theoretical work (see, e.g., Thomas & Weiss 2004) there is no generally accepted picture that can explain the formation, structuring, and dynamics of sunspots. It is generally accepted that one of the keys to increase our overall understanding of sunspots lies in a better understanding of the many observable fine-scale features. The penumbra with its convectively driven small-scale filamentary structure provides an excellent test-bed for any model of magneto-convection. A wealth of observations and modeling techniques gave us a good insight into the overall pattern and geometry of the magnetic field and plasma flows in the penumbra. The improvements in resolution and polarimetric sensitivity in recent years have revealed an increasing number of small-scale variations within the penumbral magnetic field (strength and inclination), velocity, and temperature structures (see, e.g., Schlichenmaier 2002). Recent imaging observations have even revealed fibrils of 150 to 180 km in width which contain substructures in the form of dark narrow cores, and which are less than 90 km in size (Scharmer et al. 2002). A small magnetic structuring on sub-resolution scales (perpendicular and along the LOS) has also long been suggested and recognized by the ubiquitous Stokes profile asymmetry (Solanki & Montavon 1993; Schlichenmaier et al. 2002; Müller et al. 2002; Tritschler et al. 2004; Sánchez Almeida 2005). Based on the specific signatures of unresolved magnetic field structures in the Stokes profiles, we exploit the diagnostic capa-
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abilities of the meso-structured approach proposed by Carroll & Staude (2003, 2005, 2006), which is based on a stochastic formulation of the polarized radiative transfer. This new approach is applied to spectro-polarimetric observations of a sunspot penumbra which were subject to a macroscopic 2-component inversion by Bellot Rubio, Balthasar, & Collados (2004, hereafter RBC). Using this combination of macroscopic and mesoscopic inversion allows us to estimate the characteristic length scale of the underlying magnetic fluctuation.

This paper is organized as follows: In Sect. 2 we give a brief overview of the basic concepts of the line formation in fluctuating and stochastic media with structures of finite length. In Sect. 3 we demonstrate that the statistical scattering and absorption terms of the stochastic transfer equation directly determine the degree of the area and amplitude asymmetries of the resulting Stokes-V profiles, which can be used to estimate the typical length scale of the magnetic fluctuation. In Sect. 4 we describe the basic results of the 2-component inversion and the following application of the meso-structured approach. Finally, in Sect. 5 we summarize our results.

2. The Stochastic Transfer Equation for Polarized Light

Any approach to radiative transfer modeling has to make assumptions about the underlying atmospheric structure. In atmospheres which are composed of small scale fibril structures—smaller than the resolution element—the natural question arises of how these structures can be adequately accounted for in polarized radiative transfer modeling. The two (magnetic) modeling approaches based on a macro-structured (conventional flux-tube model) or micro-structured (MISMA; Sánchez Almeida et al. 1996) description of the atmosphere cannot deal with a-priori unknow structures of finite spatial extent. To describe the polarized radiative transfer in an arbitrary fluctuating atmosphere, we pursue a rigorous probabilistic formulation of the underlying atmospheric structure. A detailed description of that stochastic approach and the derivation of the stochastic transport equation for polarized light, is given in Carroll & Staude (2005, 2006). In this contribution we will only summarize the basic concepts of this approach which puts forward the idea of a more general meso-structured magnetic atmosphere (MESMA; Carroll & Staude 2006).

A prerequisite for a stochastic modeling of structures with finite spatial extent is to take into account the complete hierarchy of spatial correlations. If we assume that the spatial coherency or correlation of the individual atmospheric components along a given trajectory can be approximated by a Markov process, all the higher spatial correlations can be reduced to first-order correlation effects. We therefore introduce a random atmospheric vector $\mathbf{B}$ which comprises all relevant atmospheric parameters such as temperature, pressure, velocity, magnetic field strength, magnetic field inclination, etc., and specify the spatial correlation of the individual atmospheric components by the following discontinuous Markov process (the space-dependent, conditional-probability density function)

$$p(B''(s + \Delta s) | B'(s)) = e^{-\gamma'(B') \Delta s} \delta(B' - B'') + \left[ 1 - e^{-\gamma'(B') \Delta s} \right] p(B'')$$  \hspace{1cm} (1)

where $\gamma'(B')$ is the fluctuation rate of the atmospheric structure $B_s$ and $p(B'')$ the spatial independent (stationary) probability density. The fluctuation rate
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can be expressed with the help of the correlation length \( l \) (characteristic length scale) to give
\[
\gamma(B_s) = l^{-1}(B_s) .
\] (2)

This discontinuous Markov process, also known under the name Kubo-Anderson process, was already used by Frisch & Frisch (1976) to describe inhomogeneous velocity fields in stellar and solar atmospheres. This process describes the spatial relationship in terms of the conditional probability density, \( p(B'', s + \Delta s \mid B', s) \). It gives the probability of being in the state or atmospheric regime \( B'' \) at the spatial position \( s + \Delta s \), after having moved a short distance \( \Delta s \) from the known atmospheric state \( B' \) at \( s \). The probability for staying in the initial regime \( B' \), expressed by the Dirac delta function, decays exponentially with \( \Delta s \) while a sudden jump into another regime \( B_{s+\Delta s} \) has an exponentially increasing probability and is only weighted by the overall stationary probability density of the final state \( B'' \). This conditional probability density function therefore describes the correlation between the two spatial positions \( s \) and \( s + \Delta s \) along a given trajectory (later the LOS). The degree of correlation is controlled by the fluctuation rate, \( \gamma \), or the correlation length, \( l \), respectively, which describe the mean path length or extent of the atmospheric structure. This generic stochastic model of the atmospheric structure allows us to formulate a differential equation for the depth-dependent atmospheric probability density function (PDF) \( p(B, s) \) from which we can derive a stochastic transport equation for the polarized light (see Carroll & Staude 2005, 2006, for a detailed derivation). For the particular application in this contribution, an appropriate model may be given by the discontinuous Kubo-Anderson process Eq. (1), which allows to derive a master-like transport equation for the mean conditional Stokes vector, which reads
\[
\frac{\partial Y_B}{\partial s} = -K_B Y_B + J_B + \int \tilde{\gamma}(B) Y_B p(B', s) \, dB' - \int \gamma(B') Y_B p(B') \, dB' ,
\] (3)

where
\[
\tilde{\gamma}(B) = \frac{p(B)}{p(B, s) l(B)} \quad \text{and} \quad Y_B(s) = \int I_p(I, s \mid B, s) \, dI ,
\]
are the modified fluctuation rate and the mean conditional Stokes vector, respectively. Please note, that \( p(B) \) and \( p(B, s) \) are in general not the same distributions, the latter being the spatial dependent PDF of the atmospheric state vector \( B \) which evolves from a given initial state along the ray path, whereas the stationary PDF, \( p(B) \), is assigned a priori by the atmospheric model. The evolution of \( p(B, s) \) is governed by a Master equation (Carroll & Staude 2005, 2006). If the atmosphere is structured on finite scales, \( p(B, s) \) eventually converges to the stationary distribution \( p(B) \), the equilibrium state.

The stochastic transport equation for the mean conditional Stokes vector has a rather simple physical interpretation. Since \( Y_B \) is conditioned on a particular atmospheric regime \( B \), and the transport equation (3) describes its statistical evolution through the atmosphere, four physical processes govern the transport of the mean conditional Stokes vector: the two processes of true absorption and thermal emission, and two processes that describe the statistical inflow and outflow of intensity to or from the regime \( B \) under consideration. In detail,
the third term on the r.h.s. of Eq. (3) gives the amount of intensity or photons which may enter from all other atmospheric regimes into $B$ and can therefore be considered as an additional (statistical) source or scattering vector, while the fourth term describes the (statistical) loss or absorption of intensity due to transitions of photons from $B$ to all other possible atmospheric regimes. The degree of statistical scattering and absorption is controlled by the correlation length $l$ of the particular atmospheric regime. The observable of our problem—the expectation value of the Stokes vector at the top of the atmosphere—can easily be obtained from a final integration of the mean conditional Stokes vector over the entire atmospheric state space of $B$,

$$
\langle I(s) \rangle = \int Y_B(s) p(B, s) \, dB .
$$

(4)

3. Stokes Profile Asymmetries

Stokes-$V$ profile asymmetries can be attributed to the existence of gradients in velocity and magnetic fields (Landolfi & Landi Degl’Innocenti 1996; López Ariste 2002). Thus the net circular polarization (NCP) and area asymmetry are good indicators for the underlying magnetic inhomogeneity along the LOS. Carroll & Staude (2006) show that in a magnetic fluctuating or randomly organized atmosphere the statistical scattering term in Eq. (3) is responsible for most of the NCP. To demonstrate this effect, we performed some simple model calculations for the Fe I line at 1564.8 nm, where we have assumed a stochastic 2-component atmosphere, which consists of a field-free ensemble of structures with a net downflow of 1.5 km/s, and a stationary magnetic ensemble of structures with a field strength of 700 G. To quantify the degree of asymmetry we used the usual definitions of the area asymmetry, $\delta A$, and the amplitude asymmetry, $\delta a$,

$$
\delta A = s \frac{\int_{\lambda} V(\lambda) \, d\lambda}{\int_{\lambda} |V(\lambda)| \, d\lambda} , \quad \delta a = s \frac{\max\{V(\lambda)\} + \min\{V(\lambda)\}}{\max\{V(\lambda)\} - \min\{V(\lambda)\}} ,
$$

(5)

where $s$ is the sign of the blue lobe of the Stokes-$V$ profile.

Figure 1 shows the Stokes-$V$ profile area and amplitude asymmetry for Fe I 1564.8 nm as a function of the correlation length. Figure 1 nicely demonstrates, how the area and amplitude asymmetries depend on the correlation length (characteristic length scale) of the underlying atmospheric structures. The smaller the correlation lengths, or higher the fluctuation rates, the stronger the asymmetries. Please note, that only the correlation length has changed in that scenario. For very small correlation lengths the asymmetries asymptotically reach the micro-turbulent limit ($l \rightarrow 0$). For a better comparison of the asymptotical behavior the true micro-turbulent limit was calculated under the MISMA approximation and is denoted by the dashed line in Fig. 1. For very large correlation lengths and low fluctuation rates (the macrostructured limit) the asymmetries asymptotically approach zero. This is the expected result as the magnetic and non-magnetic structures in our model calculations possess no intrinsic gradients. This behavior is also a direct consequence of the stochastic transport equation (3) where the statistical scattering and absorption are inversely proportional to the correlation length $l$, and in the limit $l \rightarrow \infty$ the different atmospheric
regimes finally decouple and no statistical scattering and absorption can occur. Carroll & Staude (2006) show that the stochastic transfer equation presented here includes, in fact, the micro-structured (MISMA) approximation as well as the macro-structured approach as limiting cases. Please note, that already for correlation lengths greater than 10 km significant deviations from the micro-turbulent approximation occur. As the inherent magnetic fluctuation of the atmosphere has an important effect on the degree of the NCP or area asymmetries (Carroll & Staude 2006), these asymmetries can be used in turn to estimate the underlying length scale of the magnetic fluctuation.

4. Penumbral Structures from a Stochastic 2-Component Approach

The starting point of our analysis are the results of a 2-component inversion of a sunspot penumbra from RBC. Their retrieved overall penumbral model is in agreement with a discontinuous model of the so-called uncombed penumbra proposed by Solanki & Montavon (1993), with an almost horizontal flux-tube component embedded in a more vertical background field component. The inversion were performed by a special adaptation of the SIR code (see RBC for details) and returns the temperature stratification, the three components of the single-valued magnetic field vector, and the LOS velocity of the two components of the model, together with a single value for the macro-turbulent velocity. In addition, the stray light factor $\alpha$ and the filling factor of the flux-tube component are also determined. We do not go into the details of the analysis and results of the inversion of RBC, but we want to emphasize that the quality of the fitted profiles of the 2-component inversion seems to be able to grasp the essential character of the underlying penumbral atmosphere, despite the fact that the magnetic field and velocity structures of the two components are assumed to be neither depth dependent nor interlaced. The 2-component inversion of RBC represents therefore a typical macro-structured approach, because the two components are not interlaced along the LOS and, unlike the observed Stokes-V
profiles, the resulting synthetic profiles therefore exhibit no NCP or area asymmetry. In order to allow for a more flexible meso-structured approach we have incorporated the model parameters of the 2-component atmospheres into our stochastic (meso-structured) transfer model. The underlying atmosphere in this stochastic scenario can now be considered as the composition of two different ensemble structures (the background and the flux-tube ensemble) but this time they are really interlaced. The mean length scale or extent of the individual ensemble structures is determined by the correlation length, which at the same time determines the rate of the fluctuation within the atmosphere and the ability of the atmosphere to produce NCP. For each individual average profile the correlation length of the underlying structures were adjusted until a best fit with the observation were reached. The profiles were averaged over a small portion of pixels (3 x 3). We put particular emphasis on profiles along the magnetic neutral line because of the preferred viewing angle onto the magnetic structures in this region. Figure 2 shows an averaged Stokes-V profile located in the magnetic neutral line. Despite the fact that the initial model parameters of the 2-component inversion are relatively good (Fig. 2, left), they cannot account for the area asymmetry of the observed profile. The observed Stokes-V profile of Fe I 1564.8 nm has a NCP of 4.9 mÅ and an area asymmetry of −17.6%. Since the infrared iron line is not very sensitive to velocity and magnetic-field gradients or fluctuations, due to its weakness and small range of formation, the observed profiles must be subject to a rather large fluctuation rate. After adjusting the correlation length (with all other parameters kept fixed), the fit of the profiles could be considerably improved and the resulting NCP of 4.9 mÅ and area asymmetry of −17.3% could be almost exactly reproduced. To obtain these large values of the Stokes-V asymmetries, with the given model parameters of the 2-component inversion, a correlation length as low as 40 km was needed. The analysis of a number of Stokes-V profiles along the magnetic neutral line gave similar results. For each of the analyzed Stokes-V profiles we could obtain a significantly improved fit compared to the macroscopic analysis of RBC, simply by varying the underlying correlation length. From our analysis we could infer that the typical length scale of the magnetic fluctuation in a penumbra must be between 30 and 70 km.
5. Summary

The stochastic polarized radiative transfer recently developed by Carroll & Staude (2005, 2006) was applied to a meso-structured magnetic model (MESMA; Carroll & Staude 2006) of a sunspot penumbra to estimate the characteristic length scale of the magnetic fluctuation. The degree of fluctuation directly controls the amount of the NCP (or area asymmetry) of the resulting circular polarization signal. A maximum degree of asymmetry is always produced by a micro-structured or micro-turbulent atmosphere, whereas the NCP vanishes for a macro-structured atmosphere. However, as it can be seen from Fig. 1, these two cases only represent the very far limits of a more general meso-structuring that is valid for the broad range of finite length scales. Based on the results of 2-component analysis of a sunspot penumbra (RBC), we applied a method which is able to account for the finite fluctuation of magnetic components along the LOS. With this approach, we were able to estimate the correlation length of the underlying magnetic structures, for a number of locations along the magnetic neutral line. For the characteristic length scales of the magnetic fluctuation we found values between 30 and 70 km. These findings indicate that the magnetic field in the penumbra, although it may have on average a typical uncombed-like structure, may possess a considerable amount of substructures. We conclude that the meso-structured analysis presented here, may provide a viable approach for studying small-scale and complex magnetic fields in the solar atmosphere.

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