Instabilities in the magnetic tachocline

Rainer Arlt
Astrophysikalisches Institut Potsdam, An der Sternwarte 16, D-14482
Potsdam, Germany

Abstract. The stability of the solar tachocline is reviewed with emphasis on magnetic instabilities. The combined results of the last decade show that instabilities are very likely beyond magnetic field strengths of roughly 1000 Gauss. High-latitude field belts are more easily destructed by magnetic instability. Non-linear calculations indicate though that the instabilities of low-latitude fields do not necessarily lead to fully developed turbulence and destruction of the large-scale field by turbulent diffusion.

1. The tachocline in a nutshell

The convection zone constantly generates a differential rotation as a result of rotating, stratified convection. The rotation is essentially a function of latitude – a useful approximation for the moment. Upon going deeper into the Sun, the stratification becomes subadiabatic and convection ceases. The transition is very sharp; only a few percent of the solar radius are threaded by overshooting plumes from the above convection zone near a radius of \(0.71 \, R_\odot\).

The interior could now exhibit a rotation profile close to a Taylor-Proudman state in which the angular velocity is constant on cylinders (or \(\Omega\) is a function of the axis distance \(s\) only). Helioseismology tells us, however, that the interior solar rotation is nearly uniform, and that the transition from the differential rotation in the convection zone to the uniform internal rotation is thinner than \(D = 0.05 \, R_\odot\). The abrupt change of the rotation profile has led to the term tachocline for the transition layer at about 0.67–0.72 \(R_\odot\). The term does not imply anything about the thermal stability of the layer. How much of the tachocline is convectively stable is still a matter of debate. The below considerations refer to the radiative part of the tachocline, be it the entire domain or not.

Some numbers for the quantities encountered in the radiative tachocline may be appropriate here. The temperature is a bit more than \(2 \times 10^6\) K and the sound speed is about \(2 \times 10^7\) cm/s which is more than a hundred times larger than the equatorial rotation speed in the tachocline. The density scale-height is 12% of the solar radius while the pressure scale-height is only 8% of the solar radius. The Brunt-Väisälä frequency (buoyancy frequency) is about \(N = 8 \times 10^{-4}\) s\(^{-1}\) in the middle of the tachocline, increasing with depth inside the Sun. The rotational frequency is about \(\Omega_0 = 2.7 \times 10^{-6}\) s\(^{-1}\), whence \(\Omega/N < 1\) but not necessarily negligible.

With a kinematic viscosity of \(\nu = 10 \ldots 20\) cm\(^2\)/s and a magnetic diffusivity \(\eta\) of somewhere between 500 and 2000 cm\(^2\)/s, the tachocline possesses a mag-
netic Prandtl number of roughly $Pm = \nu/\eta = 10^{-2}$. The ratio of the viscous time-scale to the rotational time-scale is expressed by the Reynolds number $Re = DR\Omega_0/\nu \approx 2 \times 10^{13} \ldots 4 \times 10^{13}$, while we will often use the magnetic Reynolds number $Rm = DR\Omega_0/\eta \approx 2 \times 10^{11} \ldots 10^{12}$ in the tachocline. While the formation and maintenance of the thin tachocline is not in the scope of this Paper, we consider the stability differential rotation alone and under the presence of magnetic fields.

Magnetic fields have not been detected by helioseismology at tachocline depth so far. The detection threshold still lies beyond 100 kG, probably at about 300 kG (Antia, Chitre, & Thompson 2000; Antia, Chitre, & Thompson 2003). Several dynamo models attempting to explain the variable activity of the sun imply strong magnetic fields at the bottom of the convection zone which rise due to instabilities, penetrate the convection zone locally and form sunspots eventually at the surface. Theoretical computations investigated the question of how strong magnetic fields have to be in the tachocline, in order to rise locally and arrive at observed heliographic latitudes at the solar surface. The thin-flux tube approximation leads to suitable fields of the order of 100 kG (e.g. Choudhuri & Gilman 1987; Granzer et al. 2000). More specifically, the observed tilt of bipolar groups can be achieved by field strengths in thin flux tubes of 60–120 kG (D’Silva & Choudhuri 1993). This is within the lower limit for helioseismic detections, but the question is whether magnetic fields can be amplified to such strong field strengths without becoming unstable in a realistic tachocline environment. We are reviewing the various results about magnetic instability of fields below the bottom of the convection zone, which are not based on the thin-flux tube approximation but on distributed fields similar to the ones resulting from mean-field dynamo models. This compilation of studies is far from complete; it is a condensed view of the facts on which investigations appear to agree, rather than a full bibliography of papers on the solar tachocline.

We try to group instabilities into categories by their energy source. We find three groups relevant for the tachocline: (i) shear-driven instabilities for which differential rotation is essential, but not necessarily the only ingredient, (ii) current-driven instabilities which will be collectively called Taylor instabilities in the following, and (iii) buoyancy-driven instabilities.

2. Shear-driven stabilities

Instabilities draining their energy from the shear are typically purely hydrodynamic; if a gradient in the mean background flow is large enough, the flow may become linearly unstable against small perturbations. Even if the shear is subcritical, finite-amplitude perturbations can cause a nonlinear instability in the flow. In the context of the tachocline, it is the latitudinal gradient in the angular velocity $\Omega(r, \theta)$ which is closest to causing an instability, where $r$ is the radial coordinate and $\theta$ is the co-latitude running from the north pole to the south pole. A finite-amplitude perturbation could be provided by the convective overshooting into the tachocline below.

Differential rotation may lead to a shear-driven instability if large enough. Watson (1981) was concerned with the latitudinal profile of the angular velocity $\Omega(\theta)$ at the bottom of the convection zone and found instability for a difference of
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the angular velocity at the equator and at the pole of more than about 29%. This result appeared to be close to the actual value in the Sun. Today we know that the radiative part of the tachocline possesses very likely a latitudinal difference of less than that. While the result was based on a differential rotation profile of the form $1 - \alpha_1 \cos^2 \theta$ where the critical $\alpha_1$ is determined, refined analyses including a fourth power, like $-\alpha_2 \cos^4 \theta$, delivered reduced stability limits of down to about tachocline values (Dziembowski & Kosovichev 1985; Charbonneau, Dikpati, & Gilman 1999). If the radial decrease of differential rotation is taken into account in a 3D model, an increase of the stability limits compared to purely latitudinal gradients is observed, apparently by weak viscous coupling of horizontal layers (Arlt, Sule, & Rüdiger 2005). We may thus conclude that the tachocline is just about stable hydrodynamically. If hydrodynamic instabilities are at work, they may exhibit only rather low saturation levels, according to Garaud (2001).

A magnetic, but still shear-driven instability is the magneto-rotational instability (MRI; Velikhov 1959; Balbus & Hawley 1991). Its importance lies chiefly in the domains of accretion disks and galactic gas, but has also been applied to stellar radiation zones for angular-momentum redistribution (Arlt, Hollerbach, & Rüdiger 2003). The criterion typically cited for the MRI is an angular velocity decreasing with axis distance. There is indeed a region in the solar tachocline which exhibits such a gradient and lies between heliographic latitudes between 40° and 95°. The prospects for MRI in the tachocline and possible implications for the dynamo were explored by Parfrey & Menou (2007).

The stable stratification of the radiative zone modifies the MRI criterion though. According to Ogilvie (2007), the MRI is not excited in the tachocline despite the fact that $\partial \Omega / \partial s < 0$. Instead, a negative gradient of $\Omega$ over co-latitude would be required, because of the strong confinement to horizontal motions.

3. Current-driven instabilities

While the magneto-rotational instability does exist for magnetic background fields without currents (homogeneous in the simplest case), magnetic-field configurations with currents provide another class of instabilities which do not require differential rotation, not even rotation at all. The first detailed studies of the instability date back to the early seventies (Vandakurov 1972; Tayler 1973), and we will refer to this class of instabilities as Tayler instability. A wide range of magnetic field configurations was found to be unstable. There is only the current-free magnetic fields, such as a homogeneous field and a toroidal field $B_\phi$ depending on the axis distance $s$ like $B_\phi \sim s^{-1}$ are stable, as are force-free fields which require a perfectly conducting exterior though (Chandrasekhar & Kendall 1957). Fields amplified by differential rotation in the tachocline are not force-free.

A current in the mean magnetic field, together with any given perturbation, causes a Lorentz force and can lead to growth of an unstable mode at some supercritical background field (even for infinitely small background fields in some cases of ideal MHD). While specific geometries of computational domains and background magnetic fields lead to a variety of stability criteria, we are most concerned with rotating systems here. An approximate criterion was found by
Figure 1. Maximum magnetic field strength in Gauss according to the estimate given by Pitts & Tayler (1985). The gray shades represent the angular velocity profile.

Pitts & Tayler (1985) for toroidal fields $\sim s$ saying that a system is Tayler stable if the Alfvén frequency, $\Omega_B = B/\sqrt{\mu \rho_0} s^2$ is smaller than the rotation frequency $\Omega$; $\mu$ and $\rho_0$ are the magnetic permeability and the density, respectively. We can thus easily plot a diagram for the Sun taking $\Omega(r, \theta)$ from helioseismology and plotting the corresponding maximum stable magnetic field. Figure 1 shows approximate angular velocity contours as well as contours of marginally stable magnetic fields in Gauss according to the Pitts & Tayler relation using the standard solar model for the density and a solar-like $\Omega(r, \theta)$. Higher angular velocities near the equator allow for stronger stable magnetic fields; denser regions can host stronger magnetic fields as well. The limit is about $2 \times 10^5$ G being consistent with what is required to form low-latitude sunspots according to thin-flux tube simulations. However, the specific geometry and stronger gradients of the magnetic field (the currents coming along with them in particular) in the tachocline and the presence of differential rotation may lead to other stability limits.

All models investigating the magnetic stability of the solar tachocline place belts of toroidal magnetic fields into the tachocline. The question is how high one can drive the field strength of the background field before instabilities set in. Perturbations are typically non-axisymmetric with low azimuthal wave numbers, since these are the most easily excited ones. The typical equations are incom-
pressible MHD such as the Boussinesq approximation

\[
\frac{\partial \vec{u}}{\partial t} = \nu \nabla^2 \vec{u} - (\vec{u} \cdot \nabla) \vec{u} - \nabla p - \alpha \Theta \vec{g} + \frac{1}{\mu \rho_0} (\nabla \times \vec{B}) \times \vec{B},
\]

(1)

\[
\frac{\partial \vec{B}}{\partial t} = \eta \nabla^2 \vec{B} + \nabla \times (\vec{u} \times \vec{B}),
\]

(2)

\[
\frac{\partial \Theta}{\partial t} = \kappa \nabla^2 \Theta - \vec{u} \cdot \nabla (\Theta + T_s),
\]

(3)

where \(\vec{u}, \vec{B},\) and \(\Theta\) are the velocity, magnetic field, and temperature (deviations). The other symbols are \(p\) for the gas pressure, \(\nu\) and \(\eta\) as above for the kinematic viscosity and magnetic diffusivity, \(g\) for the gravity, and \(T_s\) for the background temperature profile. There is no density stratification in these equations, but it can be implemented in form of the anelastic approximation as well (Miesch, Gilman, & Dikpati 2007), although the results were not found to be altered significantly.

An extensive, two-dimensional approach to the stability of toroidal magnetic fields in the tachocline is that of Gilman & Fox (1997). These fields were of broad-band type \(\sim \sin \theta \cos \theta\) and showed the typical properties of the instability of short growth-rates of the order of months beyond field strengths of several 10 kG. The critical fields strength was not accessible to that model though.

Interestingly, the instability was called joint instability by Gilman & Fox (1997) and magneto-shear instability by Cally, Dikpati, & Gilman (2003), because it only occurred as a result of the joint effect of toroidal magnetic field and differential rotation. The presence of the Tayler instability without differential rotation may be hidden though in the two-dimensional approximation used. The fact that the instability comes along with jets, that is increased latitudinal gradients in \(\Omega\), is an indication that the shear is not the primary energy source for the unstable mode. Also the energy conversion rates given by Gilman & Dikpati (2000) show a dominance of draining energy from the background magnetic field over draining the background rotation. The picture seems to reverse for high \(m\) though.
Figure 3. Stability limits for bands of toroidal magnetic fields at different latitudes in a non-ideal MHD domain according to improved computations compared to Arlt et al. (2007b). The limits are for a magnetic Reynolds number of \( R_m = 10^4 \) and a magnetic Prandtl number of \( P_m = 1 \). Flows symmetric with respect to the equator correspond to antisymmetric modes of the magnetic field and vice versa.

Still linear but three-dimensional is the attempt by Arlt, Sule, & Rüdiger (2007a). A spherical shell without simplifications in the geometry is simulated in the Boussinesq approximation. The belt of toroidal magnetic field is now a function of both latitude and radius, \( B_\phi(r, \theta) \) as is the differential rotation. The onset of the instability without shear is illustrated in the left panel of Figure 2. Magnetic field lines of the unstable \( m = 1 \) mode, which is antisymmetric with respect to the equator, are shown here. The corresponding flow is symmetric. The right panel shows the same unstable mode in a differentially rotating domain. Besides the opening of the field lines to form a clamshell type configuration, one can also see the polar kink described by Cally (2003).

Various computations address the dependence of the stability on the latitude of the magnetic field belt. Non-ideal searches for the actual stability limits by the author led to Figure 3 for \( R_m = 10^4 \) and \( P_m = 1 \). The computations confirm the result that toroidal magnetic fields are less stable at high heliographic latitudes. This would be in line with the conception that toroidal fields at high latitudes cannot provide field strengths strong enough to reach the surface when rising through the convection zone.

Typical shapes of \( B_\phi \) belts are Gaussians (Dikpati & Gilman 1999) or powers of sine-functions, such as \( \sin^{2p} \theta \cos^{2q-1} \theta \) with natural numbers \( p \) and \( q \) (Gilman & Fox 1997; Arlt, Sule, & Filter 2007b). The precise shape of the field belts do not appear to have great importance for the results, but very narrow bands may favour instability of \( m > 1 \) modes (Dikpati & Gilman 1999).

While the instability of \( m = 1 \) modes has been studied extensively, it is desirable to have a look at other \( m \). The study by Arlt et al. (2007b) again employs a three dimensional domain in the sense that all quantities depend on \( r, \theta, \) and \( \phi \) but the latter more specifically in terms of individual Fourier modes.
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Figure 4. Growth rates of individual $m$-modes for three different magnetic Prandtl numbers, $P_m = 1$, $P_m = 0.5$, and $P_m = 0.1$. The background field is fixed at $S = 2000$ and the magnetic Reynolds number is $R_m = 10^4$.

which deliver equations decoupled in $m$ in the linear treatment. The method also involved a physical viscosity and diffusivity. The stability limits will thus not only depend on the field strength of the background magnetic field, but also on the Reynolds number (or magnetic Reynolds number) and the magnetic Prandtl number. Starting with the latter, the growth rates of individual $m$-modes is plotted in Figure 4 fixing the magnetic Reynolds number and the magnetic field strength $B_0$. Assuming that the mode with the highest growth rate is also the one most easily excited (an assumption which all studies based on growth rates have to make), we see that the influence of $P_m$ is small. Since the solar value is not too far from the modelled ones, it is unlikely that the numerical difficulties with very small $P_m$ lead to wrong conclusions on the most easily excited $m$.

Diffusive codes such as the one used by Arlt et al. (2007b) face the problem that they cannot model the real solar magnetic Reynolds numbers of about $10^{12}$. This is where Figure 5 helps in showing the trend of the most easily excited $m$ towards high $R_m$. The trend points to small $m$, and it looks reasonable to consider $m = 1$ the most easily excited mode, also because the lowest $m$ invokes smaller gradients subject to diffusion inhibiting an instability. This is compatible with the analysis by Dikpati, Gilman, & Rempel (2003) who find similar growth rates for $m = 1$ and $m = 2$ in a variety of cases in ideal MHD, but significantly lower growth rates for $m = 3$ or $m = 4$. The study also nicely illustrates that the symmetry of the emerging modes is not clearly defined. It very much depends on the specific configuration whether a mode with symmetric or antisymmetric flow grows fastest, where the symmetry is meant with respect to the equator. Note that in all these cases, a symmetric flow mode corresponds to an antisymmetric mode of the magnetic field.

Most of the stability analyses were concerned with purely toroidal magnetic fields. The combination with a poloidal field, often being referred to as a twisted field, can give rise to higher stability limits (Wright, 1973). It is even possible to construct fields with unlimited stability, provided they are embedded in a
Figure 5. Growth rates of individual $m$-modes for three different magnetic Reynolds numbers, $R_m = 2000$, $R_m = 10000$, and $R_m = 14000$. Now, $P_m = 1$, and again $S = 2000$.

perfectly conducting exterior (Chandrasekhar & Kendall 1957) which is not applicable to the tachocline. The computations by Arlt et al. (2007b) also used a poloidal field $B_{pol}$ to generate a toroidal one by solar differential rotation up to $B_\phi/B_{pol} = 100$ and showed an increase of the stability limit to about 1000 Gauss when scaling the results to solar parameters.

Three-dimensional, but not specifically addressing the tachocline are the computations by Kitchatinov & Rüdiger (2008) which involve all three diffusivities at play in a full MHD problem, namely viscosity, and magnetic and thermal diffusivities. The result is reduced stability for toroidal fields down to about 600 G when applied to the upper radiation zone of the Sun.

When making statements about the real-world growth rates of the Taylor instability, one could think of the following scenario. If the magnetic field being prone to instability grows on a certain time-scale, the field will eventually reach the stability limit where the growth rate is zero. We could imagine a poloidal field which is amplified in a differential rotation. As the background field is further amplified, the instability becomes faster and will catch up with the amplification time-scale. In case the instability leads to turbulence and thus turbulent diffusion, the background field cannot grow much further. Winding up the toroidal field is a matter of a few solar rotations. A few months is therefore the minimum time-scale for the Taylor instability to become effective. Faster growth is not possible because there is no mechanism amplifying the field into the supercritical regime more efficiently than differential rotation.

Knowing about the linear stability limits of nonaxisymmetric modes for a given background magnetic field, one is tempted to ask for the nonlinear evolution of the instability. A detailed study is the one by Cally et al. (2003) employing various field belts in a nonlinear, slightly diffusive 3D system. The simulations show maximum tipping angles of the field belts of about $15^\circ$ for mid-latitude belts, before the fields decay because of lack of an energy source. Belts at lower latitudes reach tipping angles of less than $10^\circ$. These results were
confirmed by Miesch et al. (2007) in both Boussinesq and anelastic approximation. A constantly restored field by a dynamo process on top of the tachocline will be a promising subject for future simulations. The unstable modes and their nonlinear evolution provide too little kinetic helicity to provide a dynamo-$\alpha$ on their own, according to the simulations by Miesch et al. (2007).

4. Buoyancy-driven instabilities

Talking about the magnetic tachocline, we will briefly address also the question, to which extent magnetic fields can lead to buoyant instabilities despite the very stable stratification in the radiative tachocline.

Thomas & Nye (1975) derived an instability criterion for a distributed field in a local context. A vertical gradient of the magnetic field fulfilling

$$\frac{\eta}{\kappa} > N^2 \frac{\eta}{\kappa}$$

is unstable, where $c_s$ is the sound speed and $v_A$ is the Alfvén speed of the magnetic field. Taking into account that magnetic and thermal diffusivities are competing effects, it was found that the left hand-side of (4) has to be $> N^2 \eta/\kappa$ for instability, implying that the stabilizing effect of the sub-adiabatic stratification is reduced, since $\eta < \kappa$ in the tachocline (cf. Hughes 2004 for a review and references). A numerical study by Tobias & Hughes (2004) showed that shear suppresses the buoyant instability because of its tendency to "axisymmetrize" local (i.e. nonaxisymmetric) perturbations, as the authors call it.

5. Summary and outlook

The general picture created by what we described in the previous sections seems to indicate a mildly turbulent tachocline. Differential rotation alone is either stable or very slightly supercritical. If magnetic fields beyond 1 kG are generated in the tachocline, they will be subject to Tayler instability modified by shear. Both hydrodynamic and magnetic instabilities do not show the excitation of fully developed turbulence though. The storage of toroidal magnetic fields in the tachocline is definitely limited. It is interesting to note that the toroidal fields emerging in the tachocline formation scenario based on an internal fossil field are about 200 G (Rüdiger & Kitchatinov 1997) which are on the safe side with respect to the Tayler instability.

Models of thin-flux tube rise required strong fields of $10^5$ G in order to reproduce the low emergence latitudes of sunspots and the tilt angle of bipolar groups. New results by Fan (2008) for more realistic, distributed flux tubes show that both coherent rise to the surface and correct tilt angles with such tubes of $10^5$ G are not achievable. There is certainly a lot more to be explored beyond stability analyses of magnetic fields. Turbulent MHD models of flux rising through the convection zone will be a challenge for the next decade. If these will confirm the need of $10^5$ G, dynamo models and the location of the strongest fields need to be revisited.
A dynamo located in the upper part of the convection zone, as was recently receiving support from simulations by Brandenburg (2005), is certainly an option for the solar dynamo which needs to be addressed by studies, besides the tachocline-based dynamos.

If the Pitts-Taylor criterion is applicable to the tachocline, and if dynamo properties involve the tachocline instabilities, one may expect ten times stronger magnetic fields in the tachoclines of fast solar-like rotators and different observational characteristics for the dynamos of such stars. Since the tachocline instabilities provide time-scales between the rotational period and the cycle time, they may be relevant for the observed changes in zonal flows which do vary on such time-scales.

References

Dziembowski, W., & Kosovichev, A. 1987, Acta Astr., 37, 341
Thomas, J. H., Nye, A. H. 1975, Phys. Fluids, 18, 490
Vandakurov, Yu. V. 1972, Soviet Astron., 16, 265
Velikhov, E. P. 1959, Sov. Phys. JETP, 9, 995