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Suppression of the large-scale Lorentz force by turbulence

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ABSTRACT

The components of the total stress tensor (Reynolds stress plus Maxwell stress) are computed within the quasilinear approximation for a driven turbulence influenced by a large-scale magnetic background field. The conducting fluid has an arbitrary magnetic Prandtl number and the turbulence without the background field is assumed as homogeneous and isotropic with a free Strouhal number St. The total large-scale magnetic tension is always reduced by the turbulence with the possibility of a 'catastrophic quenching' for large magnetic Reynolds number Rm so that even its sign is reversed. The total magnetic pressure is enhanced by turbulence with short correlation time ('white noise') but it is reduced by turbulence with long correlation time. Also in this case the sign of the total pressure may reverse but only for special turbulences with sufficiently large St > 1.
The turbulence-induced terms of the stress tensor are suppressed by strong magnetic fields. For the tension term this quenching grows with the square of the Hartmann number of the magnetic field. For microscopic (i.e. small) diffusivity values the magnetic tension term becomes thus highly quenched even for field amplitudes much smaller than their equipartition value. In the opposite case of large-eddy simulations the magnetic quenching is only mild but then also the turbulence-induced Maxwell tensor components for weak fields remain rather small.

Key words. magnetohydrodynamics (MHD) – magnetic fields – turbulence

1. Introduction

Differential rotation and fossil fields do not coexist. A nonuniform rotation law induces azimuthal fields δB_ϕ from an original poloidal field B_R which together transport angular momentum in radial direction reducing the shear $\delta\Omega$ via the large-scale Lorentz force $\boldsymbol{J} \times \boldsymbol{B}$, i.e.

$$R\frac{\delta\Omega}{\delta t} \simeq \frac{B_R \delta B_\phi}{\mu_0 \rho \Delta R}. \tag{1}$$

As the induced B_ϕ results as $\delta B_\phi \simeq \Delta\Omega B_R \delta t$ the duration of the complete decay of the shear (i.e. $\delta\Omega = \Delta\Omega$) is $\delta t \simeq \sqrt{\mu_0\rho}R/B_R$. This is a short time of order 10,000 yr for a fossil field of 1 Gauss compared with the time scale of the star formation. All protostars should thus rotate rigidly.

Equation (1) is also used for the explanation of the observed torsional oscillations of the Sun. With $B_R \simeq 5$ Gauss and $B_\phi \simeq 10,000$ Gauss the estimation for $R\delta\Omega$ is 10 m/s which is close to the observed value of 5 m/s. The result is that – if Eq. (1) is correct – the maximal field strength of the invisible toroidal fields should not be much higher than 10,000 Gauss. However, the solar convection zone is turbulent and it is not yet clear whether Eq. (1) is also true for conducting fluids with fluctuating flows and fields.

In this paper the total Maxwell stress is thus derived for a turbulent fluid under the presence of a uniform background field $\boldsymbol{B}$. The fluctuating flow components are denoted by $\boldsymbol{u}$ and the fluctuating field components are denoted by $\boldsymbol{b}$. The standard Maxwell tensor

$$M_{ij} = \frac{1}{\mu_0} B_i B_j - \frac{1}{2\mu_0}\boldsymbol{B}^2\delta_{ij} \tag{2}$$

for the considered MHD turbulence turns into the generalized stress tensor

$$M_{ij}^{\rm tot} = M_{ij} - \rho Q_{ij} + M_{ij}^{\rm T} \tag{3}$$

with the one-point correlation tensor

$$Q_{ij} = \langle u_i(\boldsymbol{x},t) u_j(\boldsymbol{x},t)\rangle \tag{4}$$

of the flow and the turbulence-induced Maxwell tensor

$$M_{ij}^{\rm T} = \frac{1}{\mu_0}\langle b_i(\boldsymbol{x},t) b_j(\boldsymbol{x},t)\rangle - \frac{1}{2\mu_0}\langle \boldsymbol{b}^2(\boldsymbol{x},t)\rangle\delta_{ij}. \tag{5}$$

The generalized Lorentz force $\boldsymbol{F}$ is then

$$F_i = M_{ij,j}^{\rm tot}. \tag{6}$$

If the only preferred direction in the turbulence is the uniform background field $\boldsymbol{B}$ both the tensors Q_{ij} and $M_{ij}^{\rm T}$ have the same form as the Maxwell tensor (2) but with two unknown scalar parameters. It makes thus sense to write

$$M_{ij}^{\rm tot} = \frac{1}{\mu_0}(1-\kappa)B_iB_j - \frac{1}{2\mu_0}(1-\kappa_{\rm p})\boldsymbol{B}^2\delta_{ij} \tag{7}$$

for the total stress tensor (3). The first term of the RHS describes a tension along the magnetic field lines while the second term is the sum of the magnetic-induced pressures transverse to the lines of force. The main role of the first term in stellar physics is an outward-directed angular momentum transport if $B_R\, B_\phi < 0$. If its coefficient $1-\kappa$ would change its sign under the presence of turbulence then for the same magnetic geometry the angular momentum transport would be inwardly directed. The Lorentz force (6) with (7) becomes

$$\boldsymbol{F} = (1-\kappa)\ \boldsymbol{J}\times\boldsymbol{B} - \frac{1}{2\mu_0}(\kappa-\kappa_{\rm p})\ \nabla\boldsymbol{B}^2, \tag{8}$$

so that the 'laminar' Lorentz force $\boldsymbol{J}\times\boldsymbol{B}$ has to be multiplied with the factor $1-\kappa$ and an extra magnetic pressure appears if the κ's are unequal (and they are) due to the action of the turbulence. If the κ is positive then its amplitude should not exceed unity as otherwise the direction of the Lorentz force reversed.

Rüdiger et al. (1986) found κ as positive and as running with the magnetic Reynolds number Rm of the turbulence even for $\mathrm{Rm}>1$. Kleeorin et al. (1989) suggest that $\kappa_{\rm p}>0$ and even larger than unity so that the total magnetic pressure changes its sign and becomes negative. The resulting instability may produce structures of concentrated magnetic field and may be important for sunspot formation (Kleeorin et al. 1990; Brandenburg et al. 2010).

Hence, the κ's have an important physical meaning. In the simplest case they both would result as negative. Then the effective pressure is increased by the magnetic terms and also the tension term $1-\kappa$ remains positive so that the Lorentz force in turbulent media is simply amplified. Many more serious consequences would result from positive κ and $\kappa_{\rm p}$ if exceeding unity. In this case the Lorentz force changes its sign with dramatic consequences for the theory of torsional oscillations and solar oscillations (Kleeorin & Rogachevskii 1994: Kleeorin et al. 1996).

The turbulence-induced modification of the Maxwell stress can also be important in other constellations where large-scale fields and turbulence simultaneously exist. Tachocline theory, jet theory, the structure of magnetized galactic disks (Battaner & Florido 1995) or oscillations of convective stars with magnetic fields could be mentioned.

2. Equations

We apply the quasilinear approximation known also as the second order correlation approximation (SOCA). All preliminary steps to derive main equations for the problem at hand were described by Rüdiger & Kitchatinov (1990). The fluctuating magnetic and velocity fields are related by the equation

$$\hat{\boldsymbol{b}}(\boldsymbol{k},\omega) = \frac{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{B})}{-\mathrm{i}\omega+\eta k^2}\hat{\boldsymbol{u}}(\boldsymbol{k},\omega), \tag{9}$$

where the hat notation marks Fourier amplitudes, e.g.

$$\boldsymbol{b}(\boldsymbol{r},t) = \int \mathrm{e}^{\mathrm{i}(\boldsymbol{k}x-\omega t)}\hat{\boldsymbol{b}}(\boldsymbol{k},\omega)\ \mathrm{d}\boldsymbol{k}\mathrm{d}\omega, \tag{10}$$

and the same for the velocity field. The influence of mean magnetic field on turbulence is described by the relation

$$\hat{\boldsymbol{u}}(\boldsymbol{k},\omega) = \frac{\hat{\boldsymbol{u}}^{(0)}(\boldsymbol{k},\omega)}{1+\frac{(\boldsymbol{k}\cdot\boldsymbol{V})^2}{(-\mathrm{i}\omega+\eta k^2)(-\mathrm{i}\omega+\nu k^2)}}, \tag{11}$$

where $\boldsymbol{V}=\boldsymbol{B}/\sqrt{\mu_0\rho}$ is the Alfvén velocity of the large-scale background field, η is the microscopic magnetic diffusivity, and ν is the microscopic viscosity. In Eq. (11) $\hat{\boldsymbol{u}}$ is the (Fourier-transformed) velocity field modified by the mean magnetic field and $\hat{\boldsymbol{u}}^{(0)}$ stands for the velocity of the 'original' turbulence which is assumed to exist for $\boldsymbol{B}=0$. The original turbulence is assumed as statistically homogeneous and isotropic, i.e.

$$\langle\hat{u}_i^{(0)}(\boldsymbol{k},\omega)\hat{u}_j^{(0)}(\boldsymbol{k}',\omega')\rangle = \frac{E(k,\omega)}{16\pi k^2}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right) \cdot\delta(\boldsymbol{k}+\boldsymbol{k}')\delta(\omega+\omega'), \tag{12}$$

where $E(k,\omega)$ is the positive-definite spectrum function of the turbulence. Here

$$\boldsymbol{u}^2 = \int\limits_0^\infty\int\limits_0^\infty E(k,\omega)\ \mathrm{d}k\mathrm{d}\omega \tag{13}$$

defines the rms velocity $\boldsymbol{u}$ of the original turbulence. Equations (9) to (12) suffice to derive the values of the κ's.

3. Weak field

We proceed by considering special cases. For weak mean magnetic field one finds from the expressions given by Rüdiger & Kitchatinov (1990) the relation

$$\kappa = \frac{1}{15}\int\limits_0^\infty\int\limits_0^\infty \frac{Ek^2(\nu(2\eta+\nu)k^4-\omega^2)}{(\nu^2k^4+\omega^2)(\eta^2k^4+\omega^2)}\ \mathrm{d}k\mathrm{d}\omega,$$

$$\kappa_{\rm p} = \frac{1}{15}\int\limits_0^\infty\int\limits_0^\infty \frac{Ek^2(\nu(8\eta-\nu)k^4-9\omega^2)}{(\nu^2k^4+\omega^2)(\eta^2k^4+\omega^2)}\ \mathrm{d}k\mathrm{d}\omega \tag{14}$$

(see also Roberts & Soward 1975). The expressions do not have definite signs so that it remains unclear sofar whether the large-scale Maxwell stress is increased or decreased by the turbulence. Even the signs of κ and $\kappa_{\rm p}$ may depend on the spectrum of the turbulence.

The simplest case is a turbulence with a *white-noise* spectrum containing all frequencies w with the same amplitude. Here and in the following we shall use the Strouhal number St and the normalized frequency w^*

$$\mathrm{St} = \frac{u}{l_{\rm c}}\tau_{\rm c}, \qquad w^* = \frac{wl_{\rm c}^2}{\eta} \tag{15}$$

($l_{\rm c}$ correlation length, $\tau_{\rm c}$ correlation time). The turbulence frequency w^* measures the frequency of the turbulence spectrum in relation to the diffusion frequency. It is much larger than unity if the microscopic (Spitzer) diffusivity is used. The minimum value should be unity if – as it is used in large-eddy simulations – $\eta=\eta_{\rm T}\simeq wl_{\rm c}^2$. Formally, in the numerical integrations presented below the limit $w^*=0$ describes the case of the frequency spectrum as a Dirac delta function $\delta(\omega)$.

The multiplication of St and w^* provides the magnetic Reynolds number

$$\mathrm{Rm} = \frac{ul_{\rm c}}{\eta}, \tag{16}$$

where we have used the relation $\tau_{\rm c}=1/w$ as a definition of the correlation time. Then it is $w^*=\mathrm{Rm}/\mathrm{St}$.

For white noise one finds the simple results

$$\kappa = \frac{1}{15}\mathrm{Rm\ St}, \qquad \kappa_{\rm p} = -\frac{1}{15}\mathrm{Rm\ St}, \tag{17}$$

so that the κ is positive and runs with $\mathrm{Rm\ St} = u^2\tau_c/\eta$ which is the (large) ratio of the eddy diffusivity and the microscopic diffusivity hence the magnetic tension is always (strongly) reduced. On the other hand, the magnetic pressure is increased (see Eq. (7)). The negative sign of the value of κ_p excludes the possibility that the effective pressure term in Eq. (7) changes its sign so that the total magnetic pressure becomes negative. This is formally possible after (17) for the magnetic tension parameter $1-\kappa$.

The white-noise approximation, however, is not perfect. If, for example, the opposite frequency profile for very long correlation times, i.e. $E \propto \delta(\omega)$, is applied to (14) then again the κ is positive but the sign of κ_p depends on the magnetic Prandtl number

$$\mathrm{Pm} = \frac{\nu}{\eta}. \qquad (18)$$

It is thus necessary to discuss the integrals in (14) in more detail.

3.1. $\mathrm{Pm} \geq 1$

For $\mathrm{Pm} > 8$ one finds that κ_p is negative-definite for *all* possible spectral functions. The coefficient $1-\kappa_p$ of the magnetic pressure is thus positive-definite and cannot become negative. This is not true for κ. We shall show that the κ will 'almost always' be positive so that the Lorentz force term in the generalized Lorentz force expression (8) is 'almost always' quenched by the existence of the turbulence.

For $\nu = \eta$ the expressions (14) turn into

$$\kappa = \frac{1}{15}\int_0^\infty\int_0^\infty \frac{Ek^2(3\eta^2k^4-\omega^2)}{(\omega^2+\eta^2k^4)^2}\ \mathrm{d}k\,\mathrm{d}\omega,$$

$$\kappa_p = \frac{1}{15}\int_0^\infty\int_0^\infty \frac{Ek^2(7\eta^2k^4-9\omega^2)}{(\omega^2+\eta^2k^4)^2}\ \mathrm{d}k\,\mathrm{d}\omega, \qquad (19)$$

which again do not have definite signs. One can write, however, the expression for κ as

$$\kappa = \frac{1}{15}\int_0^\infty\int_0^\infty \frac{2\eta^2k^6E}{(\omega^2+\eta^2k^4)^2}\mathrm{d}\,k\,\mathrm{d}\,\omega - \\ -\frac{1}{15}\int_0^\infty\int_0^\infty \frac{\omega k^2}{\omega^2+\eta^2k^4}\frac{\partial E}{\partial\omega}\mathrm{d}\,k\,\mathrm{d}\,\omega, \qquad (20)$$

from which κ proves to be positive-definite for all spectral functions E which do not increase for increasing ω. We shall see that the positivity of κ which reduces the effectivity of the angular momentum transport is a general result of the SOCA theory.

Further simplifications can be achieved by applying the model spectrum

$$E(k,\omega) = q(k)\frac{2w}{\pi(w^2+\omega^2)} \qquad (21)$$

with

$$\int_0^\infty q(k)\ \mathrm{d}k = u^2, \qquad (22)$$

where w is a characteristic frequency of the turbulence spectrum. For $w \to 0$ (21) represents a Dirac δ-function while $w \to \infty$ gives 'white noise'. The results are

$$\kappa = \frac{1}{15\eta}\int_0^\infty \frac{w+3\eta k^2}{(w+\eta k^2)^2}q\mathrm{d}k \qquad (23)$$

and

$$\kappa_p = \frac{1}{15\eta}\int_0^\infty \frac{7\eta k^2-w}{(w+\eta k^2)^2}q\mathrm{d}k. \qquad (24)$$

Again the κ is positive-definite. Note that for $w \to \infty$ the white-noise results (17) are reproduced. In this case the κ's are running with $1/\eta$ while for $w \to 0$ the κ's are running with $1/\eta^2$. This is a basic result: for short correlation times and for long correlation times the dependence of the κ's on the magnetic Reynolds number Rm differs. With other words: for flat frequency spectra (white noise) the κ's are proportionate to Rm while for steep spectra the factor Rm^2 appears. Note that for δ-like spectral functions the numerical coefficient for κ is 0.2 while for κ_p this factor is about 0.5. One can also find these values at the ordinate of Fig. 2.

A basic difference exists for κ and κ_p, too. While the κ is positive-definite, the κ_p can change its sign. Generally it will be positive only for small w^* but it should be negative for large w^*. Already from these arguments one finds the main complication of the problem. The shape of the turbulence spectrum has a fundamental meaning for the results.

To probe these results in detail a spectral function $q(k)$

$$q \simeq \frac{2l_c}{\pi}\frac{u^2}{1+k^2l_c^2} \qquad (25)$$

is used. The integration yields

$$\kappa = \frac{1}{15}\frac{\mathrm{St\ Rm}\sqrt{w^*}(2+\sqrt{w^*})}{(1+\sqrt{w^*})^2}, \qquad (26)$$

so that

$$\kappa \simeq \begin{Bmatrix} \frac{2}{15}\mathrm{Rm\ St}\sqrt{w^*} \\ \frac{1}{15}\mathrm{Rm\ St} \end{Bmatrix} \quad \text{for} \quad w^* \begin{cases} <1 \\ >1. \end{cases} \qquad (27)$$

Hence, for steep frequency spectra (small w^*) the κ runs with $\mathrm{St}^{0.5}\ \mathrm{Rm}^{1.5}$ while for large w^* the relation is St Rm. One finds again the sensitivity of the results against the shape of the spectra. We have to distinguish two different cases. In large-eddy simulations for the effective diffusivity the relation $\eta \simeq ul_c$ is used so that $\mathrm{Rm} \simeq 1$. As in the majority of the applications also the Strouhal number St is of the same order the coefficient (26) is a small number. If for direct numerical simulations the numerical value of Rm becomes large then there is no reason that (26) remains smaller than unity.

For κ_p there is another situation. One obtains

$$\kappa_p = \frac{1}{15}\frac{\mathrm{Rm\ St}\ \sqrt{w^*}(3-\sqrt{w^*})}{(1+\sqrt{w^*})^2}, \qquad (28)$$

hence,

$$\kappa_p \simeq \begin{Bmatrix} \frac{1}{5}\mathrm{Rm\ St}\ \sqrt{w^*} \\ -\frac{1}{15}\mathrm{Rm\ St} \end{Bmatrix} \quad \text{for} \quad w^* \begin{cases} <9 \\ >9. \end{cases} \qquad (29)$$

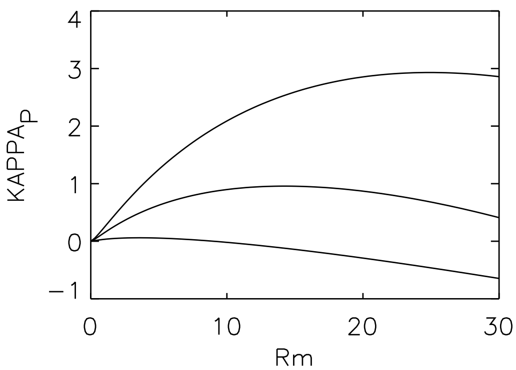

Fig. 1. The $\kappa_{\rm p}$ vs. Rm after (28). From top to bottom: St = 7, St = 4, St = 1. All curves have a maximum. Note that all $\kappa_{\rm p}$ become negative for sufficiently large Rm. Pm = 1.

For $w^* > 9$ the $\kappa_{\rm p}$ is *negative* so that the total pressure is always positive. For $w^* < 9$, however, the $\kappa_{\rm p}$ becomes positive. In this case for large Strouhal number the total magnetic pressure $1 - \kappa_{\rm p}$ becomes negative. Figure 1 demonstrates that $\kappa_{\rm p}$ exceeds unity for St > 4. For St > 4 one finds $\kappa_{\rm p} > 1$ for Rm = 14, i.e. $w^* = 3.5$. We have to stress, however, that the SOCA approximation only holds if not both the quantities St and Rm simultaneously exceed unity. For turbulences in liquid metals in the MHD laboratory Rm $\simeq 1$ is a typical value.

3.2. Pm $\ll 1$

The situation is more clear for small magnetic Prandtl numbers, which exist, e.g., in stellar interiors, protoplanetary disks and also in the MHD laboratory. It is possible to consider the limit $\nu \to 0$ in the equations (14) but only for flat turbulence spectra with finite values of the correlation time. Stationary patterns with $E \propto \delta(\omega)$ are excluded. In the limit of very small Pm the Eqs. (14) reduce to

$$\kappa = \frac{\pi}{15\eta}\int\limits_0^\infty E(k,0)\ {\rm d}k - \frac{1}{15}\int\limits_0^\infty\int\limits_0^\infty \frac{E(k,\omega)k^2}{\omega^2+\eta^2k^4}{\rm d}k{\rm d}\omega,$$

$$\kappa_{\rm p} = \frac{4\pi}{15\eta}\int\limits_0^\infty E(k,0)\ {\rm d}k - \frac{9}{15}\int\limits_0^\infty\int\limits_0^\infty \frac{E(k,\omega)k^2}{\omega^2+\eta^2k^4}{\rm d}k{\rm d}\omega. \qquad (30)$$

The spectrum (21) leads to

$$\kappa = \frac{1}{15\eta w}\int\limits_0^\infty \frac{2\eta k^2+w}{\eta k^2+w}\ q(k)\ {\rm d}k,$$

$$\kappa_{\rm p} = \frac{1}{15\eta w}\int\limits_0^\infty \frac{8\eta k^2-w}{\eta k^2+w}\ q(k)\ {\rm d}k. \qquad (31)$$

Again the κ is positive-definite. If the wave number spectrum has only a single value then

$$\kappa = \frac{1}{15}\frac{\rm Rm^2}{w^*}\frac{2+w^*}{1+w^*},$$

$$\kappa_{\rm p} = \frac{1}{15}\frac{\rm Rm^2}{w^*}\frac{8-w^*}{1+w^*}. \qquad (32)$$

The limit $w^* \to 0$ is here not allowed. Again the $\kappa_{\rm p}$ is positive (negative) for small (large) w^*. Formally, the Strouhal number St does not appear. Replacing the w^* by Rm/St in both limits the $|\kappa_{\rm p}|$ runs linearly with St Rm, i.e. with $1/\eta$.

The κ also runs with $1/\eta$ even for large w^*, i.e.

$$\kappa \simeq \frac{1}{15}{\rm St\ Rm} \qquad (33)$$

while for small w^* the value is $\kappa \simeq (2/15)$St Rm. For large w^*, i.e. for Rm > St, and Pm $\lesssim 1$ there is practically no influence of the numerical value of the magnetic Prandtl number (see Eq. (27)). Below we shall also demonstrate by numerical solutions of the integrals that Eq. (33) forms the main result of the present analysis. Whether the κ-coefficient may become larger than unity only depends on the numerical values of St and Rm. For large-eddy simulations with St = Rm = Pm = 1 the κ is basically only of order 0.1.

With the spectral function (25) the results are very similar, i.e.

$$\kappa_{\rm p} = \frac{1}{15}\frac{\rm Rm^2}{w^*}\frac{8-\sqrt{w^*}}{1+\sqrt{w^*}}. \qquad (34)$$

The expression (34) does only exceed unity for St $\gg 1$. For small St the sum $1-\kappa_{\rm p}$ is thus always positive independent of the actual value of Rm contribution. number Pm.

4. Strong fields

Sofar only the influence of weak magnetic fields has been considered. The influence of strong magnetic fields is also important to know. The rather complex results of the SOCA theory with arbitrary magnetic field amplitudes and with free values of both diffusivities are given in the Appendix. These expressions can be discussed by applying the single-scale wave number spectrum

$$q(k) = 2u^2\delta(k - l_{\rm c}^{-1}) \qquad (35)$$

and the frequency spectrum (21). Such an approximation allows to solve the equations (A1)...(A4) numerically including the frequency integration so that the turbulence quantities $\kappa/{\rm Rm}^2$ and $\kappa_{\rm p}/{\rm Rm}^2$ only depend on the Lundquist number

$${\rm S} = \frac{Bl_{\rm c}}{\sqrt{\mu_0\rho}\eta} \qquad (36)$$

of the magnetic field, the frequency w^* and the magnetic Prandtl number Pm. In the weak-field limit, S $\ll 1$, one finds the overall result that $\kappa/{\rm Rm}^2$ runs as $1/15w^*$ (Figs. 2, 3, top) so that again the general result (33) is reproduced. For very small w^*, i.e. for delta function frequency spectra (or, what is the same, for very long correlation times), the κ's run with $1/{\rm Rm}^2$ – as already shown above.

When the field is not weak, the stress parameters rapidly decrease with S. Figures 2 and 3 demonstrate that the magnetic quenching can be written as

$$\kappa \simeq \frac{\kappa_0}{1+\epsilon\ {\rm S}^2} \qquad (37)$$

(see Fig. 4), in opposition to Brandenburg et al. (2010) who found a cubic magnetic quenching power. From the Figures one finds that $\epsilon \lesssim 1$ for Pm < 1. For large Pm the ϵ is

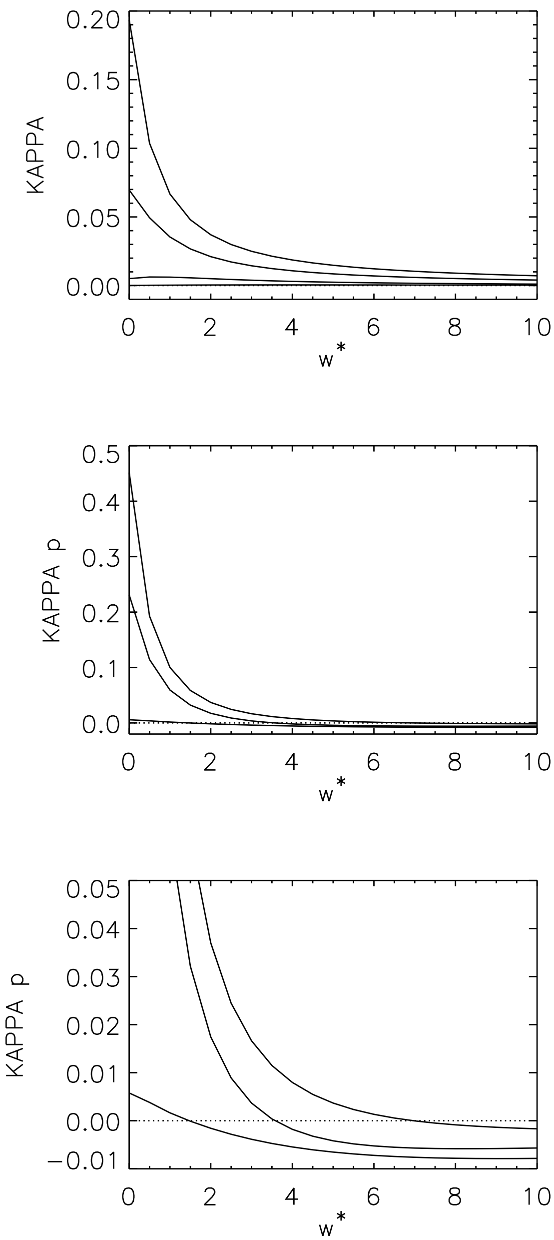

Fig. 2. The κ/Rm^2 (top) and $\kappa_\mathrm{p}/\mathrm{Rm}^2$ (middle, bottom) vs. w^* for the one-mode model (35). The curves in the plots from top to bottom are for S = 0.01, S = 1, S = 3 and S = 10. At the vertical axis the values are valid for the delta function spectra. The quantities vanish as $1/w^*$ for $w^* \to \infty$ leading to the result (33). Bottom: Details for $\kappa_\mathrm{p}/\mathrm{Rm}^2$. Pm = 1.

even smaller. The magnetic quenching of the κ-parameter is thus stronger for small magnetic Prandtl number than for large Pm. While a magnetic field with S = 1 reduces the κ remarkably if Pm < 1 in the opposite case Pm > 1 the κ is almost uninfluenced by S = 1. Figure 5 demonstrates the inverse dependence of the ϵ on the magnetic Prandtl number. One finds $\epsilon \simeq 0.75/\mathrm{Pm}$. The quenching expression, therefore, turns for Pm $\neq 1$ into

$$\kappa \simeq \frac{\kappa_0}{1+0.75\ \mathrm{Ha}^2} \tag{38}$$

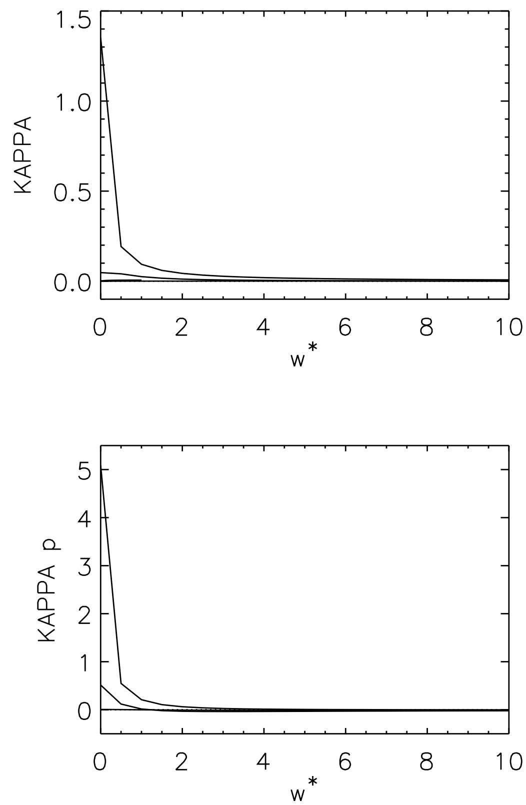

Fig. 3. The same as in Fig. 2 for Pm = 0.1. From top to bottom S = 0.01, S = 1, S = 3 and S = 10. The quantities vanish as $1/w^*$ for $w^* \to \infty$.

with the Hartmann number Ha $= \mathrm{S}/\sqrt{\mathrm{Pm}}$ instead of the Lundquist number S. For the magnetic quenching it is thus not important which of the diffusivities is large and which is small. The quenching is very strong if one of them is small. For the high-conductivity limit ($\eta \to 0$) or for inviscid fluids ($\nu \to 0$) the Hartmann number Ha takes very large values so that even very weak fields strongly suppress the κ-effect.

Note that

$$\mathrm{S} = \mathrm{Rm}\frac{B}{B_\mathrm{eq}} \tag{39}$$

with $B_\mathrm{eq} = \sqrt{\mu_0 \rho\ \langle u^2 \rangle}$ as the equilibrium field strength. The magnetic quenching of the κ-term thus grows with Rm^2 (Brandenburg & Subramanian 2005) so that for growing Rm the κ becomes smaller and smaller:

$$\kappa \simeq \frac{1}{15\epsilon}\frac{\mathrm{St}\ B_\mathrm{eq}^2}{\mathrm{Rm}\ B^2}. \tag{40}$$

It becomes thus clear that in the high-conductivity limit even for rather small fields the κ-term in Eq. (7) takes very small values which do not play an important role in the mean-field magnetohydrodynamics. Indeed, in the numerical simulations by Brandenburg et al. (2010) one finds a massive magnetic quenching of the κ's already far below the equipartition value $B = B_\mathrm{eq}$.

On the other hand, to model large-eddy simulations use Rm = 1 so that the well-known standard expression

$$\kappa = \frac{\kappa_0}{1+\epsilon\frac{B^2}{B_\mathrm{eq}^2}} \tag{41}$$

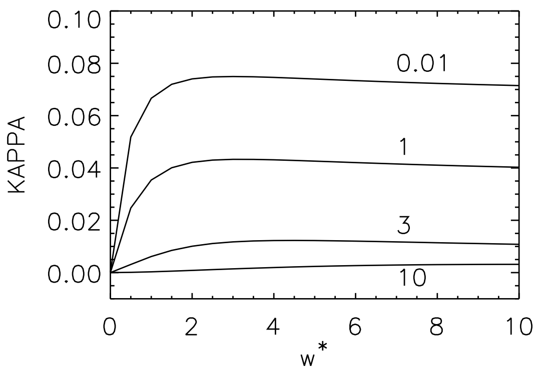

Fig. 4. The verification of the relation (38) for the functions $w^*\kappa$ marked by their Lundquist numbers S. The resulting value for ϵ is about 0.75. Pm = 1.

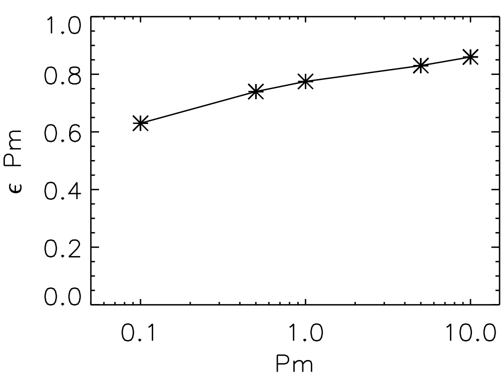

Fig. 5. The (weak) dependence the quantity $\mathrm{Pm}\cdot\epsilon$ on the magnetic Prandtl number Pm.

for magnetic quenching appears with ϵ of order unity slightly differing for small and large w^*.

Because of St = Rm = 1 in this case the κ's always remain smaller than unity in accordance to (33). Hence in both the possible concepts, i.e. the use of the microscopic diffusivities and the use of the large-eddy simulations with subgrid diffusivities, the values of the turbulence-induced Maxwell tensor coefficients remain small.

5. Catastrophic quenching?

We have computed the stress tensor which is formed by large-scale background fields, by the Reynolds stress of a turbulence field under the influence of the field and the turbulent Maxwell stress of the field fluctuations. All contributions can be summarized in form of the classical Maxwell stress tensor but with turbulence-modified coefficients (see Eq. (7)). The modified pressure term is now $1-\kappa_{\rm p}$ while the modified magnetic tension term is written as $1-\kappa$. The quantities κ and $\kappa_{\rm p}$ have been computed within the quasilinear approximation (SOCA) which can be used if the minimum of both the numbers St and Rm is (much) smaller than unity. As almost all turbulences fulfill the condition $\mathrm{St}\simeq 1$, the validity of SOCA requires $\mathrm{Rm}=ul_c/\eta<1$. Under this restriction the resulting κ's are always smaller than unity. For all magnetic Prandtl numbers Pm we found κ as positive so that the non-pressure force term $(\boldsymbol{B}\nabla)\boldsymbol{B}$ is reduced under the influence of turbulence. This is in particular true for the coefficients of the angular momententum transport terms $B_\phi\ B_R$ and $B_\phi\ B_z$ which, therefore, become more and more ineffective in turbulent fluids.

The sign of $\kappa_{\rm p}$ strongly depends on the magnetic Prandtl number. It proves to be negative-definite for large Pm. For smaller Pm the sign of $\kappa_{\rm p}$ depends on the shape of the frequency spectrum of the turbulence. For steep profiles, i.e. very long correlation times, the $\kappa_{\rm p}$ becomes positive while for flat frequency-spectra of the turbulence which are as flat as the spectrum of white noise (very short correlation times) the $\kappa_{\rm p}$ for Rm < 1 becomes negative.

One could believe that relations valid for small Rm like $\kappa_{\rm p}\propto \mathrm{St}\cdot\mathrm{Rm}$ can be also used for Rm > 1 so that finally the effective magnetic pressure becomes negative. This, however, is not true. The $\kappa_{\rm p}$ changes its sign for $\mathrm{Rm}\gg\mathrm{St}$ and becomes negative. Hence, the total magnetic pressure results as mostly positive. The only exception exists for sufficiently large St and sufficiently small Rm (see Fig. 1).

More dramatic is the situation with the magnetic tension and its coefficient $1-\kappa$ which is also the coefficient of the vector $\boldsymbol{J}\times\boldsymbol{B}$ in the generalized Lorentz force in turbulent media. This coefficient is positive for small κ, i.e. for sufficiently small Rm if St = 1. It is positive and smaller than unity for the large-eddy simulations ('mixing-length model') considered at the end of Sect. 3.2 with Rm = St = Pm = 1 (see Fig. 2).

The question, however, whether the κ can exceed unity (so that $1-\kappa$ becomes negative) cannot finally be answered within the quasilinear approximation. It is $\kappa\simeq 0.1\ \mathrm{St}\cdot\mathrm{Rm}$ where one of the factors St and Rm *must* be smaller than unity but the product $\mathrm{St}\cdot\mathrm{Rm}$ is formally not restricted by the SOCA. It is thus a clear and surprising result also in the frame of SOCA that the angular momentum transport by large-scale magnetic fields can strongly be *suppressed* under the influence of turbulence. The possible existence of $\kappa>1$ has been confirmed by the numerical simulations by Brandenburg et al. (2010).

The formal background of this phenomenon is that the integrals defining κ and $\kappa_{\rm p}$ do not exist in the high-conductivity limit or, what is the same, in the ideal MHD. The same is true for the much simpler magnetic-suppression problem of the eddy diffusivity. We take the expression

$$\eta_{\rm T}=\frac{1}{3}\int\limits_0^\infty\int\limits_0^\infty \frac{\eta k^2\ E}{\omega^2+\eta^2k^4}\left(1-\frac{6}{5}\frac{\eta^2k^4-\omega^2}{(\omega^2+\eta^2k^4)^2}\ \frac{\boldsymbol{B}^2}{\mu_0\rho}\right){\rm d}k\,{\rm d}\omega. \quad (42)$$

(Kitchatinov et al. 1994) for the SOCA expression of the eddy diffusivity under the presence of a uniform magnetic background field (Pm = 1). The expression is part of a series expansion which converges if the second term is smaller than the first term. The second term of the RHS of this expression has two important properties: i) it is positive for all spectral functions E with $\partial E/\partial\omega<0$ so that the $\eta_{\rm T}$ is always reduced by the magnetic fields and ii) it does not exist for the limit $\eta\to 0$. With other words, for rather small η the integral becomes large so that the magnetic quenching would be extremely effective for large Rm. This is why such a series expansion only holds for very weak fields. This phenomenon has been called a 'catastrophic' quenching (see Blackman & Field 2000; Blackman & Brandenburg 2002). It exists within the SOCA theory for the eddy diffusivity and also for the eddy viscosity. One finds from Eq. (42) that the mentioned diffusivities are magnetically quenched like $1-\mathrm{S}^2$ for small S and like S^{-3} for large S. Of course, by this procedure the η_T cannot become negative. We know, on the other hand, that the magnetic quenching of the eddy diffusivity in sunspots reduces its value (only) from $5\cdot 10^{12}$ cm^2/s to about 10^{11} cm^2/s what – together with the time decay law of the sunspots – can be understood with quenching expressions like (42) for Rm = 1 (Rüdiger & Kitchatinov 2000). It is thus suggested to work with the simple relations Rm = 1 and $\mathrm{S} \simeq B/B_\mathrm{eq}$ in applications with turbulent convection.

Similarly, also the κ increases for vanishing η. There is, however, no nonmagnetic term against which the magnetic influence can be neglected as it must be compared with the large-scale Lorentz force $\boldsymbol{J} \times \boldsymbol{B}$ which is also of the second order in $\boldsymbol{B}$. The only possibility to keep the turbulence contribution small for large Rm is to put St $\ll 1$. However, if the magnetic field is super-equipartitioned then the κ is magnetically quenched which introduces a new factor Rm^{-2}. Then the magnetic-induced κ-effect finally runs with 1/Rm so that it vanishes in the high-conductivity limit. In summary, for large Rm and for very weak magnetic field the κ can exceed unity (so that the stress tensor reverses sign) but this phenomenon disappears already for rather weak fields.

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Appendix A: SOCA expressions of the κ's

The expressions for the mean-field Lorentz force parameters κ and κ_p of Eq. (7) provided by the quasilinear theory for arbitrary magnetic amplitudes can be written as

$$\kappa = \int_0^\infty \int_0^\infty \frac{E(k,\omega)k^2}{\omega^2+\eta^2 k^4}\ K(B,k,\omega)\ \mathrm{d}k\mathrm{d}\omega,$$

$$\kappa_\mathrm{p} = \int_0^\infty \int_0^\infty \frac{E(k,\omega)k^2}{\omega^2+\eta^2 k^4}\ K_\mathrm{p}(B,k,\omega)\ \mathrm{d}k\mathrm{d}\omega. \tag{A.1}$$

The kernel functions K and K_p depend on the magnetic field and the variables k and ω via

$$\beta = \frac{kV}{\left(\omega^2+\eta^2 k^4\right)^{1/4}\left(\omega^2+\nu^2 k^4\right)^{1/4}},$$

$$LN = \log\left(\frac{\beta^2 - 2\beta\sin\frac{\phi}{2}+1}{\beta^2+2\beta\sin\frac{\phi}{2}+1}\right),$$

$$AR = \arctan\left(\frac{\beta-\sin\frac{\phi}{2}}{\cos\frac{\phi}{2}}\right) + \arctan\left(\frac{\beta+\sin\frac{\phi}{2}}{\cos\frac{\phi}{2}}\right) \tag{A.2}$$

with $V = \eta\mathrm{S}/l_\mathrm{c}$. Here,

$$\cos\phi = \left(\eta\nu k^4 - \omega^2\right)/\sqrt{(\omega^2+\eta^2k^4)(\nu^2k^4+\omega^2)} \tag{A.3}$$

The kernels read

$$K = \left(\frac{\omega^2+\eta^2k^4}{\omega^2+\nu^2k^4}\right)^{1/2}\frac{1}{8\beta^4}\left(-(\beta^2+3)\frac{LN}{4\beta\sin\frac{\phi}{2}} + (\beta^2-3)\frac{AR}{2\beta\cos\frac{\phi}{2}}\right) + \frac{1}{8\beta^4}\left(6-(\beta^2-3+6\cos\phi)\frac{LN}{4\beta\sin\frac{\phi}{2}} - (\beta^2+3+6\cos\phi)\frac{AR}{2\beta\cos\frac{\phi}{2}}\right) \tag{A.4}$$

and

$$K_\mathrm{p} = \left(\frac{\omega^2+\eta^2k^4}{\omega^2+\nu^2k^4}\right)^{1/2}\frac{1}{4\beta^4}\left(\frac{8}{3}\beta^2+(\beta^2-1)\frac{LN}{4\beta\sin\frac{\phi}{2}} - (\beta^2+1)\frac{AR}{2\beta\cos\frac{\phi}{2}}\right) + \frac{1}{4\beta^4}\left(2-(\beta^2-1+2\cos\phi)\frac{LN}{4\beta\sin\frac{\phi}{2}} - (\beta^2+1+2\cos\phi)\frac{AR}{2\beta\cos\frac{\phi}{2}}\right). \tag{A.5}$$

The first parts in these expressions represent the contribution of the Reynolds stress while the next two lines represent the small-scale Maxwell stress.