The turbulent EMF as a time series and the ‘quality’ of dynamo cycles

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Following earlier suggestions to replace the ensemble average used in the mean-field electrodynamics by an averaging over the azimuthal coordinate we consider the basic coefficients in the turbulent electromotive force (EMF) as time-dependent functions. The well-known coefficients $\alpha$ and $\eta_T$ – both in the relevant tensorial formulations – are derived from one and the same turbulence field with maximal helicity so that in a local formulation the total turbulent EMF is described as a time series. The (kinematic) turbulence models have always the same intensity of $u'$ (rms) $\simeq$ 100 m s$^{-1}$ and the number of the eddies in the unit length is varied. Both the EMF coefficients $\alpha$ and $\eta_T$ are evaluated within the limit of high (microscopic) conductivity.

Both coefficients prove to exhibit time series with remarkable fluctuations. The fluctuations are stronger for the $\alpha$-effect compared with the eddy diffusivity, and they are stronger if the number of cells is decreased. In general, we find fluctuations dominating the average for turbulence with only a few but large cells. Even changes of the sign of the EMF coefficients occur for short periods.

Application of the resulting turbulence EMF-coefficients to an 1D $\alpha^2\Omega$-dynamo model leads to complicated time series for the resulting magnetic field. It is oscillatory for an infinite number of cells and becomes more complex the less turbulence eddies are operating in the flow. For decreasing eddy population the corresponding spectral line in the power spectrum of the magnetic cycles becomes more and more broad (the ‘quality’ of the cycle sinks) – but further reduction of the cell population even leads to a chaotic character of the dynamo amplitude. Finally, the difference between oscillatory and stationary solutions of the dynamo model seems to disappear. The observed quality of the solar cycle might be produced by about 100 giant cells along the equator.
1 Motivation

It seems as would Earth and Sun realize two different forms of magnetic dynamos. While the solar magnetic activity exhibits a distinct periodicity with a period of 11 years, the terrestrial magnetic field is ‘permanent’. Indeed, two different sorts of dynamos in spherical geometry have been constructed with the desired properties. The so-called $\alpha^2$-dynamo yields stationary solutions while the $\alpha\Omega$-dynamo yields oscillatory solutions even (under certain assumptions with the correct period (cf. Krause and Rädler, 1980; Rüdiger and Brandenburg, 1995). Observation and theory are not very far from correspondence. There are, however, striking differences in the time behavior: the solar activity is not strictly periodic and the Earth dynamo is not strictly permanent. Known are the variations of the solar cycle between 9 and 13 yrs, also known is the Maunder minimum status for the Sun (Nesme-Ribes et al., 1994) as well as for several stars (Balintas and Vaughan, 1985; Balintas et al., 1995). The power spectrum of the solar cycle does not form a delta function. The quality $\omega/\Delta\omega$ of the 11-year cycle seems to be of order of 5 (Wittmann, 1978; Hoyng, 1993).

Also the Earth magnetic field varies. There are irregular reversals of the field strength, the shortest period of stability was 40 000 yrs. The average length of the intervals of constancy is almost 10 times larger than this minimum value (cf. Krause and Schmidt, 1988). Recently such a reversal has numerically been simulated with a 3D MHD code by Glatzmaier and Roberts (1995).

The source of the irregularities has been sought in the nonlinearities of the problem which appear if the magnetic fields are no longer weak. In this case, e.g., the $\alpha$-effect is quenched by the magnetic field, but also the diffusivity and/or the differential rotation are magnetically influenced. And, in particular, the resulting large-scale magnetic field forms a Lorentz force driving a large-scale ‘Malkus-Proctor’ flow which itself influences the magnetic field (Malkus and Proctor, 1975).

There are several papers with the motivation to explain Grand minima such as the Maunder minimum by various nonlinearities (Weiss et al., 1984; Brandenburg et al., 1989; Brandenburg et al., 1990; Jennings and Weiss, 1991; Kitcatinov et al., 1994; Tobias, 1996) in the mean-field equations. The averages are taken over an ‘ensemble’, i.e. a great number of identical examples.

The other possibility to explain the temporal irregularities considers the characteristic turbulence values as a time series. The idea is that the average procedure only concerns a periodic spatial coordinate, e.g. the azimuth $\phi$. In other words, in a development after Fourier modes such as $e^{im\phi}$ the mode $m = 0$ is considered as the mean value. Again, if the time scale of this mode does not vary during the correlation time local formulations such as in (6) are reasonable. Hence, in the high-conductivity limit

$$\tau_{\text{corr}} |\langle \dot{B} \rangle| \ll |\langle B \rangle|$$

(1)
must hold. Nevertheless, the turbulence intensity, the $\alpha$-effect and the eddy diffusivity become time-dependent quantities (Hoyng, 1988; Choudhuri, 1992; Hoyng, 1993; Moss et al., 1992; Hoyng et al., 1994).

Questions here concern the amplitude, the time scales as well as the phase relation between (say) helicity and eddy diffusivity. Is the effect strong enough for significant influence on the dynamo? For a discussion of such questions we imagine a random turbulence model as given and, in a certain approximation, we evaluate the complete turbulent electromotive force (EMF) in a consistent way as a time series. In our final section the consequences of this concept are computed for a simple plane-layer dynamo with differential rotation. The main parameter which we have to vary shall be the number of the cells in the unit length, here defined as the total longitudinal domain.

2 Basic equations

Basic quantity in the mean-field electrodynamics is the mean electromotive force

$$\mathbf{E} = \langle \mathbf{u}' \times \mathbf{B}' \rangle,$$  \hspace{1cm} (2)

whose $\mathbf{curl}$ determines the temporal development of a mean magnetic field $\langle \mathbf{B} \rangle$:

$$\frac{\partial \langle \mathbf{B} \rangle}{\partial t} = \mathbf{curl} \ (\mathbf{E} + \langle \mathbf{u} \rangle \times \langle \mathbf{B} \rangle).$$  \hspace{1cm} (3)

In these equations the notation $\langle \rangle$ means quantities averaged by a procedure fulfilling the Reynolds rules (cf. Moffatt, 1978). $\mathbf{F}'$ denotes the difference between $\mathbf{F}$ and $\langle \mathbf{F} \rangle$. The average is usually an ensemble average. An ensemble is formed by an infinite number of copies. In this case the Reynolds rules hold, i.e.

$$\langle \langle \mathbf{F} \rangle \rangle = \langle \mathbf{F} \rangle, \quad \langle \mathbf{F} + \mathbf{G} \rangle = \langle \mathbf{F} \rangle + \langle \mathbf{G} \rangle, \quad \langle \langle \mathbf{F} \rangle \langle \mathbf{G} \rangle \rangle = \langle \mathbf{F} \rangle \langle \mathbf{G} \rangle,$$  \hspace{1cm} (4)

with $\mathbf{G}$ as another fluctuating field.

Instead of averaging over an ensemble one can also define mean values by integration over (say) space. Of particular interest is here an averaging procedure over longitude, i.e.

$$\langle \mathbf{F} \rangle = \frac{1}{2\pi} \int F d\phi$$  \hspace{1cm} (5)

(Braginskii, 1965; Choudhuri, 1992; Hoyng, 1993; Hoyng et al., 1994). Here the turbulent EMF for a given position in the meridional plane will form a time series with the correlation time $\tau_{\text{corr}}$ as a characteristic scale. The peak-to-peak variations in the time series should depend on the number of cells. They remain finite if the number of cells is restricted – as it is in reality. Only for an infinite number of the turbulence cells the peak-to-peak variation in the time series goes to zero, but it will grow for a smaller number of cells along the unit length, i.e. the circle $\phi = 0..2\pi$.  

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The local formulation of the mean-field EMF,
\[ \mathcal{E}_i = \alpha_{ij} \langle B_j \rangle + \beta_{ijk} \langle B_j \rangle_k, \quad (6) \]

is adopted with the tensors \( \alpha \) and \( \beta \). They are formed by the most simple tensorial elements such as \( (G\Omega)\delta_{ij} \) or \( \epsilon_{ijk} \) which by themselves are pseudo-tensors as it must be. \( \Omega \) is the basic rotation while \( G \) denotes the gradient of scalars such as density and/or turbulence intensity. The representation also implies that the \( \alpha \)-effect only exists for rotating and stratified turbulences, not however, the diffusivity which yields from any turbulence field. If the turbulence by itself is helical then its correlation tensor contains a pseudo-scalar (e.g. the mean helicity) even without rotation and an \( \alpha \)-effect will exist.

Both coefficients, \( \alpha \) and \( \beta \), are outgrowths from one and the same turbulence field. It is the concept, therefore, to define a helical turbulence existing in a Cartesian box and to compute simultaneously the related components of both the tensors \( \alpha \) and \( \beta \).

We restrict ourselves to the computation of the turbulent EMF in the high-conductivity limit. Then the second-order correlation approximation yields the expression
\[ \mathcal{E} = \int_0^\infty \langle u'(x,t) \times \text{curl} \left( u'(x,t-\tau) \times \langle \mathbf{B}(x,t) \rangle \right) \rangle \, d\tau, \quad (7) \]

which for \((1)\) can be written in the form \((6)\). The detailed expressions are given in Krause and Rädler (1980). We apply them for a Cartesian geometry where \( y \) represents the azimuthal direction over which the average will be taken. Of relevance for the dynamo are only the components \( \mathcal{E}_x \) and \( \mathcal{E}_y \).

It is convenient to write \((7)\) in the local form \((6)\). In components it reads
\[ \mathcal{E}_x = \alpha_{xx} \langle B_x \rangle + \beta_{zz} \frac{\partial \langle B_y \rangle}{\partial z}, \]
\[ \mathcal{E}_y = \alpha_{yy} \langle B_y \rangle - \beta_{zz} \frac{\partial \langle B_z \rangle}{\partial z}. \quad (8) \]
\( \beta_{zz} \) plays the role of a common eddy diffusivity. From \((7)\) one can read
\[ \alpha_{xx} = \int_0^\infty \langle u_y(t) \frac{\partial u_z(t-\tau)}{\partial x} - u_z(t) \frac{\partial u_y(t-\tau)}{\partial x} \rangle \, d\tau, \]
\[ \alpha_{yy} = \int_0^\infty \langle u_z(t) \frac{\partial u_x(t-\tau)}{\partial y} - u_x(t) \frac{\partial u_z(t-\tau)}{\partial y} \rangle \, d\tau, \]
\[ \beta_{zz} = \eta_T = \int_0^\infty \langle u_z(t) u_z(t-\tau) \rangle \, d\tau. \quad (9) \]
3 The turbulence model

Our numerical model studies the time evolution of the EMF coefficients (9) generated by turbulent gas motions. All the numbers used concern the solar convective zone. We analyze a parcel of the solar gas permanently perturbed by vortices distributed at all angles. At every time step the resulting velocity field is used to calculate the coefficients (9). A similar kinematic model of turbulent motions has been previously

used to evolve a small-scale magnetic field in galaxies (Otmianowska-Mazur and Urbanik, 1994). The local coordinate system has the $xy$-plane parallel to the solar equator, the $x$-axis pointing away from the center of the Sun (parallel to the solar radius), the $y$-axis tangential to the azimuthal direction and the $z$-axis directed to the south pole.

A single vortex has the form of a rotating column of gas moving with velocity $u_z$ up and down along its axis of rotation. The angular velocity of rotation $\omega$ as well as the velocity $u_{z\text{loc}}$ decrease with the distance $r$ from its rotational axis according to

$$\left(\omega, u_{z\text{loc}}\right) = \left(\omega_0, u_0\right) \cdot e^{-0.5 \left(\frac{r}{l_{\text{corr}}}^2 + \frac{z_{\text{loc}}}{z_{\text{corr}}}^2\right)},$$

where $l_{\text{corr}}$ is the adopted vortex radius (Otmianowska-Mazur et al., 1992), $z_{\text{loc}}$ is its local length parallel to the axis of rotation and $z_{\text{corr}}$ is its length scale in this direction. All velocities are truncated at $3l_{\text{corr}}$ and $3z_{\text{corr}}$. Due to Coriolis force and the density stratification the solar turbulence possesses a helical character, so in our model all vortices have rotation in the sense of a right-handed screw. It is, therefore, a turbulence now with maximal helicity. The resulting $\alpha$-effect never exceeds the rms velocity – as it must be. Every vortex has its own lifetime $t$ which decreases exponentially with the decay time $\tau_{\text{corr}}$ according to $t \cdot \exp(-t/\tau_{\text{corr}})$. A uniform distribution of vortices in 3D space is approximated with 12 possible inclination angles of the rotational axes to the $xy$-plane.

Figure 1: The velocity field for model $B$ in the plane $z = 0$. White color is for positive $v_z$ and black color is for negative $v_z$. 


At the beginning a certain number of turbulent cells starts to move. Their inclination and position in the \(xy\)-plane are chosen randomly. After an assumed period of time (one or more time steps), a fraction of them is changed to new ones, with randomly given position, inclination and with lifetime starting from zero. Physically, it means that the new source of turbulence appears somewhere in the neighborhood. The situation repeats itself continuously as time evolves.

The turbulence intensity and the other mean coefficients are calculated as a time series of space values, according to (9), where \(t\) means a given moment of time and averaging in space (angular brackets) is taken only along the \(y\)-axis. This assumption decreases computational precision, but allows us to diminish operational memory needed for simulations.

Table 1: Input and output for the turbulence models \(A, B, C\) and \(D\). \(N\) is the eddy population of the equator, \(R_{\text{tur}}\) the number of new eddies per time step, other quantities and normalizations as explained in the text.

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<tr>
<th>(A)</th>
<th>(B)</th>
<th>(C)</th>
<th>(D)</th>
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<tbody>
<tr>
<td>(l_{\text{corr}})</td>
<td>(\tau_{\text{corr}})</td>
<td>(N)</td>
<td>(R_{\text{tur}})</td>
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<tr>
<td>1.</td>
<td>10.</td>
<td>200</td>
<td>200</td>
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<tr>
<td>2.</td>
<td>20.</td>
<td>100</td>
<td>7</td>
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<tr>
<td>4.</td>
<td>40.</td>
<td>50</td>
<td>0.25</td>
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<tr>
<td>8.</td>
<td>80.</td>
<td>25</td>
<td>0.007</td>
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The computations are performed in 33, 200 and 33 grid points, in the \(x, y\) and \(z\)-axis, with a grid interval of \(\delta x = \delta y = \delta z = 2.5 \cdot 10^9\) cm = 25 Mm in each direction yielding \(5 \cdot 10^{11}\) cm along the \(y\)-axis (the length of the solar equator) and \(7 \cdot 10^{10}\) cm in the next two directions. A time step of \(\delta t = 25 \cdot 10^8\) s is adopted. In the vortex center, the gas velocity \(u_0\) and the angular velocity \(\omega_0\) are 100 m s\(^{-1}\) and 6.8 \(\cdot 10^{-6}\) s\(^{-1}\), respectively. We use normalized units allowing the use of normalized grid intervals such as \(dx = dy = dz = dt = 1\). The normalized velocity \(du\) must be calculated from \(u_0 = du \cdot \delta x / \delta t\) so that the value of \(du\) is 0.1 for a desired \(u_0 = 100\) m s\(^{-1}\). The vortex radius \(l_{\text{corr}}\) as well as the decay time \(\tau_{\text{corr}}\) and the number of the new turbulent cells \(R_{\text{tur}}\) are varied for four cases given in the Tables 1 and 2 in normalized units.

We compute turbulences with vortices inclined at 12 angles to the \(xy\)-plane (uniformly distributed in a sphere). The chosen inclination angles are given in cylindrical coordinates where \(\theta\) is the angle between the \(z\)-axis and the axis of rotation of a single vortex, \(\phi\)-the positional angle of this axis in the \(xy\)-plane. We choose 6 possible pairs of angles \(\theta\) and \(\phi\) but the flow can go up and down along their axis of rotation of the vortex resulting in 12 different directions. The pairs of angles are as follows: \((\theta = 0^\circ, \phi = 0^\circ), (\theta = 45^\circ, \phi = 0^\circ), (\theta = 45^\circ, \phi = 120^\circ), (\theta = 45^\circ, \phi = 120^\circ), (\theta = 0^\circ, \phi = 120^\circ), (\theta = 0^\circ, \phi = 120^\circ)\).
Table 2: The ratio $S$ of fluctuations and mean values

<table>
<thead>
<tr>
<th>model</th>
<th>$S_{\eta T}$</th>
<th>$S_x$</th>
<th>$S_y$</th>
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<tbody>
<tr>
<td>$\mathcal{A}$</td>
<td>0.45</td>
<td>1.03</td>
<td>0.72</td>
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<tr>
<td>$\mathcal{B}$</td>
<td>0.62</td>
<td>2.13</td>
<td>1.24</td>
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<td>$\mathcal{C}$</td>
<td>0.82</td>
<td>4.10</td>
<td>2.79</td>
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<tr>
<td>$\mathcal{D}$</td>
<td>1.27</td>
<td>8.94</td>
<td>6.62</td>
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$240^\circ, (\theta = 90^\circ, \phi = 0^\circ), (\theta = 90^\circ, \phi = 90^\circ)$. The ratio of the space ($r_{\text{corr}}$, $z_{\text{corr}}$) and time ($\tau_{\text{corr}}$) scales are of order of the value of the mean turbulent velocity. In our experiments the following values are adopted for the vortex radius $l_{\text{corr}}$ and for the length scale $z_{\text{corr}}$ in the $z$-direction in normalized units: 1, 2, 4 and 8. For successive length scales we choose the time scales $10, 20, 40$ and $80$, respectively. The ratio of both scales is always $l_{\text{corr}}/\tau_{\text{corr}} \approx 0.1$. To obtain the expected mean turbulent velocity value of 0.1 the number of new vortices $R_{\text{tur}}$ is also varied (Table 1).

4 Numerical experiments

The results are given as a time series of the turbulence intensity, the eddy diffusivity and the dynamo coefficients and as a standard deviation $\sigma$ from their averaged in time mean value

$$\sigma = \sqrt{E[(X - E(X))^2]},$$

where $E(X)$ is an averaged value of a random variable $X$. The standard deviation is computed to give information how big the fluctuations are in comparison with the mean (in time) value of a given quantity. Table 1 gives the input model parameters $l_{\text{corr}}$, $\tau_{\text{corr}}$ and $R_{\text{tur}}$ and the resulting values of coefficients. The ratio of the standard deviations to their mean values, $S$,

$$S_\eta = \sigma(\eta)/ | \eta |$$

is given in Table 2.

Let us start with the case where a single vortex has the shortest lifetime and space dimension. Figure 2 presents the results for a model with $l_{\text{corr}} = 1$ and $\tau_{\text{corr}} = 10$. All computed quantities are visible as a time series, each in its own normalized units according to their definitions. The three EMF coefficients as well as the turbulence intensity show fluctuations around their mean values averaged in time. The diffusion coefficient $\eta_T$ possesses positive values during most of the time. There are also regions with negative $\eta_T$, but only during a small fraction of the time. The negative values are not significant, because the very resulting means of
Figure 2: Time series for turbulence intensity $\sqrt{\langle u'^2 \rangle}$, eddy diffusivity $\eta_T$ and alpha-tensor components $\alpha_{xx}$ and $\alpha_{yy}$ for model $A$. Cgs units for time, velocity ($u$ and $\alpha$) and diffusivity result after multiplication with $2.5 \cdot 10^4 \text{s}, 10^5 \text{ cm s}^{-1}$ and $2.5 \cdot 10^{14} \text{ cm}^2 \text{s}^{-1}$, respectively.
Figure 3: Time series for turbulence intensity $\sqrt{\langle u'^2 \rangle}$, eddy diffusivity $\eta_T$ and alpha-tensor components $\alpha_{xx}$ and $\alpha_{yy}$ for model $D$. Cgs units for time, velocity ($u$ and $\alpha$) and diffusivity result after multiplication with $2.5 \cdot 10^4 \, \text{s}$, $10^5 \, \text{cm} \, \text{s}^{-1}$ and $2.5 \cdot 10^{14} \, \text{cm}^2 \, \text{s}^{-1}$, respectively.
The time series become more and more dominant with a decreasing number of eddies. Both the quantities happen for short periods.

For all our models the fluctuations for the $\alpha$-effect exceed those of the eddy diffusivity. The latter proves to be more stable than the $\alpha$-effect against dilution of the turbulence. This is a confirmation for those papers in which exclusively an $\alpha$-effect time series in the dynamo computations is used.
5 A plane-layer dynamo

There is a detailed literature about the influence of stochastic $\alpha$-fluctuations upon dynamo-generated magnetic fields for various models. Choudhuri (1992) discussed a very simple plane-wave dynamo basically in the linear regime. The fluctuations adopted are weak ($\approx 10\%$). While in the $\alpha\Omega$-regime the oscillations are hardly influenced, the opposite is true for the $\alpha^2$-dynamo. In the latter regime the solution suffers dramatic and chaotic changes even for rather weak disturbances. In the region between the both regimes the remaining irregular variations are suppressed in a model with nonlinear feedback formulated as a traditional $\alpha$-quenching.

Intermittency and Grand minima did not appear for supercritically and stochastically disturbed nonlinear full-sphere dynamos (Moss et al., 1992). There are interesting effects concerning parity changes but a phase-amplitude relation did not appear.

Hoyng (1993), Hoyng et al. (1994) and Ossendrijver et al. (1996) focused attention to a Waldmeier-type law according to which the weaker (stronger) cycles last longer (shorter) than average. Or - with $P_{\text{cyc}}$ as the cycle length and $R_{\text{max}}$ as the sunspot number at maximum, $\partial P_{\text{cyc}}/\partial R_{\text{max}} < 0$. The importance of such a regular behavior among the random character of the solar cycle is underlined. As the only one so far, these models have no difficulties producing the anticorrelation - if the cycle amplitude is considered as a measure for the sunspot number.$^1$ Our models will not be in accordance with such a anticorrelation law.

For simplicity, a plane 1D $\alpha^2\Omega$-model is used to illustrate the influence of the fluctuating turbulence coefficients to a mean-field dynamo It is not an overshoot dynamo. The plane has infinite extent in the radial ($x$) and the azimuthal ($y$) direction, the boundaries are in the $z$-direction. Hence, we assume that the magnetic field components in both azimuthal direction, $B$, and radial direction, $\partial A/\partial z$, depend on $z$ only. Note that $z$ points opposite to the latitude $\theta$.

The $\alpha$-tensor has only one component given by

$$\hat{\alpha} = -\alpha_0(t) \sin \pi \frac{z}{R},$$

where the lower and upper boundary shall be located at (the south pole) $z = 0$ and (the north pole) $z = 2R$, respectively. The $\alpha$-effect vanishes at the equator $z = R$ and it is assumed here to vanish also at the poles. The magnetic feedback is considered as a conventional $\alpha$-quenching $\alpha = \dot{\alpha}_c(B_{\text{tot}})$ with $\dot{\alpha}_c(B_{\text{tot}}) = (1 + B_{\text{tot}}^2)^{-1}$ and $B_{\text{tot}}^2 = B^2 + (\partial A/\partial z)^2$. The diffusivity $\eta$ is spatially uniform.

The normalized dynamo equations are derived in the same way as in Rüdiger

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et al. (1994),

\[
\frac{\partial A}{\partial t} = C_\alpha \dot{\alpha}(\zeta, t) \psi(B_{*0})B + \dot{\eta}(t) \frac{\partial^2 A}{\partial \zeta^2},
\]

\[
\frac{\partial B}{\partial t} = C_\alpha \frac{\partial}{\partial \zeta} \left( \dot{\alpha}(\zeta, t) \psi(B_{*0}) \frac{\partial A}{\partial \zeta} \right) - C_\Omega \frac{\partial A}{\partial \zeta} - \frac{\partial}{\partial \zeta} \left( \dot{\eta}(t) \frac{\partial B}{\partial \zeta} \right)
\]

with \( \zeta = \pi z/2R \). The boundary conditions are \( B(0) = B(\pi) = 0 \) and \( \partial A/\partial \zeta|_{\zeta=0} = \partial A/\partial \zeta|_{\zeta=\pi} = 0 \) confining the magnetic field normal to the boundary planes.

The influence of the \( \alpha \)-effect is controlled by \( C_\alpha \). Positive \( C_\alpha \) describes a positive \( \alpha \) in the northern hemisphere and a negative one in the southern hemisphere. \( C_\Omega \) defines the amplitude of the differential rotation, positive \( C_\Omega \) represent positive shear \( \partial \Omega/\partial r \). The dynamo operates with a positive \( \alpha \)-effect in northern hemisphere opposite to the often discussed overshoot dynamo (Schmitt and Schüssler, 1989; Rüdiger and Brandenburg, 1995).

Here, in contrast to earlier papers, both \( \alpha \)-effect and eddy diffusivity are time-dependent functions derived from one and the same random turbulence. We shall also compare our dynamo models with the above time-dependent turbulence EMF with a model operating with time-averaged quantities, i.e.

\[
\ddot{\alpha}(z) = \frac{1}{T} \int_{t-T}^{t+T} \dot{\alpha}(z, t') dt', \quad \dot{\eta}(t) = \frac{1}{T} \int_{t-T}^{t+T} \dot{\eta}(t') dt'.
\]  \hspace{1cm} (15)

Our dynamo numbers are \( C_\alpha = 5 \) and \( C_\Omega = 200 \). The turbulence models \( A, B \) and \( D \) are applied, flow patterns with small \( (A) \), medium \( (B) \) and with very large \( (D) \) eddies are used.

The dynamo model without EMF-fluctuations \( (T \to \infty, \text{`model O'}) \) induces a magnetic dipole field oscillating with a (normalized) period of 1.46 (Figure 4a). Turbulence model \( B \) also produces an oscillating dipole, but with a much more complicated temporal behavior (Figure 4e). It is not a single oscillation, the power spectrum forms a very broad line with substructures. The ‘quality’,

\[
Q = \frac{\omega}{\Delta \omega},
\]  \hspace{1cm} (16)

of this line with \( \Delta \omega \) as its half-width proves to be about 2.9. This value close to the observed quality of the solar cycle is here produced by a turbulence model with about 100 eddies along the equator (see Ossendrijver et al., 1996). There are also remarkable variations of the magnetic cycle amplitude.

As expected, the magnetic data of the turbulence field \( D \) lead to a highly irregular temporal behavior. Its power spectrum peaks at several periods (Figure 4d). The overall shape of the spectrum, however, does not longer suggest oscillations. The power of the lower frequencies is strongly increased, the high-frequency decrease varies as \( \omega^{-5/3} \) like the Kolmogorov spectrum indicating the existence of chaos.
Figure 4: Dynamo-induced magnetic toroidal fields for the turbulence models $\mathcal{O}, \mathcal{A}, \mathcal{B}$ and $\mathcal{D}$. LEFT: time series, RIGHT: power spectrum
Figure 5: High-resolution plot of the time series for turbulence models $B$ and $D$
Hence, what we generally observe is how the ‘dilution’ of turbulence is able to transform a single-mode oscillation (for very many cells) to an oscillation with low quality (for moderate eddy population) and finally to a temporal behavior close to chaos (for very few cells). It might thus easily be that the observed ‘quality’ of the solar cycle indicates the finite number of the giant cells driving the large-scale solar dynamo. The temporal behavior of the solar rotation law should be another output of such a cell number statistics. It should yield an independent test of the presented theory leading to the same number of eddies contributing to the turbulent angular momentum transport.

Figure 5c yields a high-resolution plot of the time variation of the toroidal field amplitude induced by the turbulence with only 25 eddies. We find long periods of the same magnetic field polarity followed by sudden reversals. The original oscillation period of the dynamo appears only very rarely. The difference between both oscillatory and steady solutions seems to be reduced. In this respect there is an interesting numerical experiment. Smoothing the time series of the turbulence model \( D \) with a filter-interval \( T \) of 1 diffusion time, we still observe a remarkable time-dependence of the EMF coefficients. The dynamo feels the changes of the turbulence coefficients just on the same time scale (Figure 5d). As long as the smoothed \( \alpha \)-effect is positive the resulting magnetic field is stationary rather than oscillatory. During this time the dynamo number changes its sign and the dynamo starts another regime with a different temporal behavior.

With the same procedure the turbulence field \( B \) yields almost time-independent \( \alpha \)-values (Figure 5b). Correspondingly, the dynamo approaches a regular status known from the conventional mean-field dynamo theory. The latter, therefore, proves to be the correct formulation for turbulence with many and small eddies. In the opposite case of only ‘occasional’ turbulence a nontrivial time series for both the turbulence EMF-coefficients and the magnetic field is an unavoidable consequence.

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References


